Optimization of a variable mouth acoustic horn

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SUMMARY

By using boundary shape optimization on the end part of a semi-infinite waveguide for acoustic waves, we design transmission-efficient interfacial devices without imposing an upper bound on the mouth diameter. The boundary element method solves the Helmholtz equation modeling the exterior wave propagation problem. A gradient-based optimization algorithm solves the resulting least-squares problem and the adjoint method provides the necessary gradients. The results demonstrate that there appears to be a natural limit on the optimal mouth diameter. Copyright © 2010 John Wiley & Sons, Ltd.

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1. Introduction

A recurrent theme when managing acoustic (or electromagnetic) waves is the need for *interfacial devices*, situated between a source, such as a waveguide or a transducer, and another device or surrounding space. Such devices can be designed to affect all aspects of the wave motion: the *generation* of waves, the *transport* of waves, and the *distribution* of waves toward prescribed regions away from the device. For example, the mouthpiece of a brass instrument and the internal structure of a compression driver for a loudspeaker horn aid in the *generation* of waves; exhaust mufflers for internal combustion engines and microwave feeder horns for reflector antennas control the *transport* of the wave motion; so-called constant directivity loudspeakers horns as well as arrays of loudspeaker horns and acoustic lenses are used to guide the *distribution* of sound power towards areas of interest [5].

Acoustic horns are interfacial devices that can be used for all these purposes. An example is the flared bell of a brass instrument. In addition to aiding in the radiation of sound, the bell is needed to make the instrument playable; a carefully designed flaring causes the frequencies of the lowest generated natural modes inside the instrument to approximate a harmonic series [12]. Loudspeaker systems intended for large halls and for outdoor use also

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employ acoustic horns, particularly in the high-frequency audio range. The function of the loudspeaker horn is both to concentrate the sound energy towards regions of interest and, perhaps less obviously, to increase the efficiency of the transducer located in the horn throat. The acoustic properties of the horn are quite sensitive to details in the design if the horn is large enough compared to the wavelength. In principle there is an infinite number of possible shapes, thus the payoff is potentially large to use a systematic approach, instead of just trial and error, to search for a design that best meet a set of prescribed properties of the system.

In this article, we explore such a systematic approach in order to design, using numerical optimization, an interfacial device of the acoustic-horn type. Consider a narrow (with respect to the wavelength) circular-cylindrical waveguide that exits in free space. A propagating wave that approaches the waveguide exit will partly be radiated towards the far field and partly reflected back into the waveguide. In order to control these reflections, we let a fixed portion at the end of the waveguide be subject to design: the cross-sectional radius as a function of axial position will be manipulated in order to affect the reflection properties, or equivalently, the *input acoustic impedance* of the device. In the case when such a device acts as a part of a brass instrument, the reflections should be appropriately determined in order to obtain an approximately harmonic spectrum as described above. On the other hand, for a device acting as a part of a loudspeaker system, it makes sense to minimize the reflections back into the waveguide in order to maximize the transmission efficiency of the system. The latter case is the one considered in this article.

A particular feature with the current study is that the mouth diameter of the device, that is, the diameter of the opening towards free space, will *not* be specified a priori; the mouth diameter will be a result of the optimization. Another particular feature is the use of the Boundary Element Method (BEM) for the evaluation of the pressure field in combination with the use of adjoint equations to evaluate objective-function gradients. The adjoint-equations approach makes it feasible to use gradient-based optimization with a large number of design variables, since the computational complexity of the gradient evaluation then becomes essentially independent of the number of design variables.

Henwood [13] and Morgans et al. [14] also employ the BEM in order to optimize the shape of acoustic horns. They optimize the horn with respect to its directivity properties and not, as we do here, its transmission properties. Their horn shape parameterization is based on a single cubic spline and a single Bezier curve element, respectively, which means that the number of design variables is two. This choice severely limits the class of possible shapes but allows the use of exhaustive-search algorithms such as global gradient-free optimization methods.

Shape optimization of acoustic horns with respect to its transmission properties have been studied by Bängtsson et al. [1] and Udawalpola and Berggren [17]. The design of horn-like interfacial devices was also considered by Wadbro and Berggren [18, 19] and Sigmund and Jensen [16], but using a completely different approach, namely the so-called material distribution approach to topology optimization. In that approach, the subject of optimization is the distribution of two materials, representing sound-hard solid material and air, respectively, within a region in space. The authors of the current article have recently combined shape and topology optimization in order to design an acoustic horn–lens combination that simultaneously is able to efficiently transmit and evenly distribute the sound energy within the operational frequency range [20].

The above mentioned contributions by Bängtsson et al. [1] and Udawalpola and Berggren [17] are the ones that are closest in spirit to the current study. Similarly as here, they optimize

Figure 1. Profile of the initial waveguide. To improve its radiation properties, we displace the (light-gray) end-part of the waveguide.

the transmission properties using gradient-based optimization and utilize adjoint equations to compute the gradients. However, they employ the finite element method (FEM)—instead of the BEM as here—to solve the governing Helmholtz equation. A complicating factor when using finite-element discretizations in the context of shape optimization is the use of deforming volume meshes, which often cause robustness issues and lead to complex and interdependent software components. The BEM offers a clear advantage in this respect, since only the boundary of the domain needs to be discretized. In fact, it would be quite difficult, due to practicalities with the deforming mesh, to study the current case of a variable-mouth device using the finite-element discretization.

Finally, for completeness we also like to point out a class of problems related to the one we study, namely the optimization of interfacial devices for electromagnetic waves. In particular, several authors have used numerical design optimization of horn antennas [6, 7, 9, 10, 21].

2. Problem Statement

2.1. Governing Equations

We consider a cylindrical symmetric waveguide, as illustrated in Figure 1, as the basis for optimization. The first section of the waveguide (with length d) remains fixed, while the cross sectional radius of the second (with length l) is allowed to vary during optimization.

We consider a planar wave of amplitude A that propagates from left to right (*right going wave*). At the end of the waveguide the propagating wave will partly be radiated towards the far-field and partly reflected back into the waveguide. The reflected wave travels from right to left (*left going wave*).

The inner radius a of the waveguide is chosen so that all non-planar modes are geometrically evanescent in the frequency band we are interested in. The distance d is chosen to ensure that non-planar waves are negligible at the beginning of the waveguide. We assume that all waves satisfy the Sommerfeld radiation condition at infinity.

In the exterior domain Ω (Figure 1), the acoustic pressure P is governed by the wave equation

$$\frac{\partial^2 P}{\partial t^2} = c^2 \Delta P, \quad (1)$$

where c is the speed of sound. By using the ansatz $P(x, t) = \Re \{p(x)e^{i\omega t}\}$, where $\Re$ is the

real part, we seek time harmonic solutions of the wave equation for the complex amplitude function p and angular frequency ω , which, together with appropriate boundary conditions, yields the following *state equation*

$$\Delta p + k^2 p = 0 \quad \text{in } \Omega, \quad (2a)$$

$$ikp + \frac{\partial p}{\partial n} = 2ikA \quad \text{on } \Gamma_{\text{in}}, \quad (2b)$$

$$\frac{\partial p}{\partial n} = 0 \quad \text{on } \Gamma \setminus \Gamma_{\text{in}}, \quad (2c)$$

$$\lim_{|x| \rightarrow \infty} |x| \left(\frac{\partial p}{\partial |x|} + ikp \right) = 0 \quad \text{uniformly in all directions}, \quad (2d)$$

where $k = \omega/c$ is the wave number. Boundary condition (2b) specifies, at boundary Γ_{in} , a right going wave with amplitude A and absorption of the reflected left going wave. The sound hard boundaries are described with boundary condition (2c) and (2d) is the Sommerfeld radiation condition.

2.2. Objective

The aim of the optimization is to obtain a device that efficiently transmits the incoming wave at Γ_{in} to the surroundings within a pre-specified frequency band. To measure transmission efficiency, we observe the *reflection coefficient* at Γ_{in} . The reflection coefficient is defined as

$$R = \frac{\langle p \rangle_{\text{in}} - A}{A}, \quad (3)$$

where A is the amplitude of incoming wave and $\langle p \rangle_{\text{in}}$ is the average pressure at Γ_{in} , that is

$$\langle p \rangle_{\text{in}} = \frac{1}{|\Gamma_{\text{in}}|} \int_{\Gamma_{\text{in}}} p \, d\Gamma, \quad (4)$$

where p is the solution to equation (2) and $|\Gamma_{\text{in}}|$ is the area of Γ_{in} . We minimize the reflection coefficient R at boundary Γ_{in} to maximize the transmission efficiency of the horn.

2.3. Design Variations and Sensitivities

We manipulate the design boundary Γ_d within an admissible design space $\mathcal{U}$ in order to achieve desired transmission properties. In order to perform a calculus of variation on Γ_d , we introduce a smooth vector field $\delta\varphi$ on $\mathbb{R}^3$ such that $\delta\varphi = 0$ on $\Gamma \setminus \Gamma_d$. Now we define a parameterized family of deformed boundaries $\Gamma_d(s, \delta\varphi) \in \mathcal{U}$ as

$$x_s = x + s \delta\varphi(x) \quad (5)$$

where $x_s \in \Gamma_d(s, \delta\varphi)$, $x \in \Gamma_d$, and parameter s is in an interval around zero. We denote the family of domains that corresponds to deformed boundaries $\Gamma_d(s, \delta\varphi)$ by $\Omega(s, \delta\varphi)$. The state equation (2) is solved on the deformed domain and the resulting solution is denoted by p_s . The directional derivative $\delta R(\delta\varphi)$ is defined as

$$\delta R(\delta\varphi) = \left. \frac{d}{ds} R(s, \delta\varphi) \right|_{s=0}, \quad (6)$$

where $R(s, \delta\varphi)$ is the reflection coefficient (3) for the domain $\Omega(s, \delta\varphi)$.

By differentiating the variational formulation of equation (2) and expression (3), the following expression for the directional derivative of R with respect to $\delta\varphi$ can be derived:

$$\delta R(\delta\varphi) = k^2 \int_{\Gamma_d} n \cdot \delta\varphi q p \, d\Gamma - \int_{\Gamma_d} n \cdot \delta\varphi \nabla q \cdot \nabla p \, d\Gamma, \quad (7)$$

where q is the solution to the adjoint PDE

$$\Delta q + k^2 q = 0 \quad \text{on } \Omega, \quad (8a)$$

$$ikq + \frac{\partial q}{\partial n} = \frac{1}{|\Gamma_{\text{in}}|A} \quad \text{on } \Gamma_{\text{in}}, \quad (8b)$$

$$\frac{\partial q}{\partial n} = 0 \quad \text{on } \Gamma \setminus \Gamma_{\text{in}}, \quad (8c)$$

$$\lim_{|x| \rightarrow \infty} |x| \left(\frac{\partial q}{\partial |x|} + ikq \right) = 0 \quad \text{uniformly in all directions.} \quad (8d)$$

A detailed derivation of expression (7) in a similar case can found in Bångtsson et al. [1]. Comparing equations (2) and (8), we see that

$$q = \frac{1}{2ikA^2|\Gamma_{\text{in}}|} p. \quad (9)$$

Thus, for this particular case, we do not need to solve a separate adjoint equation, but can simple write gradient expression (7) as

$$\delta R(\delta\varphi) = \frac{1}{2ikA^2|\Gamma_{\text{in}}|} \left(k^2 \int_{\Gamma_d} n \cdot \delta\varphi p^2 \, d\Gamma - \int_{\Gamma_d} n \cdot \delta\varphi \nabla p \cdot \nabla p \, d\Gamma \right). \quad (10)$$

2.4. Integral Equation Formulation

An integral formulation of the governing equation (2) is

$$\begin{aligned} \frac{1}{2}p(x) + ik \int_{\Gamma_{\text{in}}} p(y) G_x(y) \, d\Gamma(y) + \\ \int_{\Gamma} p(y) \frac{\partial G_x(y)}{\partial n_y} \, d\Gamma(y) = 2ikA \int_{\Gamma_{\text{in}}} G_x(y) \, d\Gamma(y), \end{aligned} \quad (11)$$

where $x \in \Gamma$ such that Γ is smooth at x and

$$G_x(y) = \frac{e^{-ik|x-y|}}{4\pi|x-y|}$$

is the fundamental solution to the Helmholtz equation in three dimensions.

Note that to evaluate expression (10), we need to know both the pressure and the gradient of the pressure at the design boundary. The numerical solution of integral equation (11) provides the pressure. To obtain an expression for the gradient of the pressure, we first observe that boundary condition (2c) and the assumption of rotational symmetry implies that the only nonzero component of ∇p on Γ_d is the tangential derivative in the direction orthogonal to the

Figure 2. Intermediate shape. The function β represents the perpendicular distance from reference boundary Γ_d^{ref} to current design Γ_d .

circumferential direction. By differentiating equation (11) at $x \in \Gamma_d$ in the tangential direction τ_x , we obtain the expression

$$\frac{\partial p(x)}{\partial \tau_x} = \mathcal{I}(p, x, \tau_x), \quad (12)$$

where

$$\mathcal{I}(p, x, \tau_x) = 2ik \int_{\Gamma_{\text{in}}} (2A - p(y)) \frac{\partial G_x(y)}{\partial \tau_x} d\Gamma(y) - 2 \int_{\Gamma} p(y) \frac{\partial^2 G_x(y)}{\partial \tau_x \partial n} d\Gamma(y). \quad (13)$$

Thus, by replacing p in the right-hand side of (12) with a suitable numerical approximation, we obtain numerical approximations also of the derivatives in expression (10).

2.5. Parameterization and smoothing

The choice of parameterization is important in shape optimization. The optimization algorithm can only produce shapes within the limits of the chosen parameterization. Thus, the optimal shape typically changes with the parameterization. If we directly use β , the perpendicular distance from the reference shape (Figure 2) as our design variable, we will typically obtain wiggly shapes similarly as reported by Bängtsson et al. [1].

In our numerical experiments, we start with the waveguide and let it evolve to form a suitable shape. We use two parameterization methods capable of handling free endpoints. Both methods define β as the solution to a 1D initial value problem with homogeneous initial conditions at the horn throat. We introduce a new design variable η and let β be defined as the solution of the ordinary differential equation

$$\beta'' = \eta \text{ in } \Gamma_d^{\text{ref}}, \quad (14)$$

where $\beta = \beta' = 0$ at the horn throat, for parameterization of *type 1* and

$$\beta' = \eta \text{ in } \Gamma_d^{\text{ref}}, \quad (15)$$

where $\beta = 0$ at the horn throat, for parameterization of *type 2*. We impose the constraint $\eta \geq 0$ for both parameterization methods. With this constraint, parameterization of type 1 produces convex horns and parameterization type 2 produces horns with monotonically increasing flare profiles. The terms *convex horn* and *increasing horn* are henceforth used synonymously to horns of type 1 and type 2, respectively.

The back side of the horn remains straight throughout the optimization. The distance, in the direction normal to the reference shape, from the current design to the back of the horn remains fixed at the throat and mouth.

3. The Discrete Problem

3.1. Discretization

The Boundary Element Method (BEM) numerically solves integral equation (11). The horn surface is discretized with N conical axisymmetric elements Γ_m , $m = 1, \dots, N$. We use the C^0 collocation method, that is, we assume that the acoustic pressure is constant throughout each element. The boundary-element discretization of integral equation (11) yields the linear system

$$\left(\frac{1}{2}\mathbf{I} + \mathbf{M} + ik\mathbf{L}\right)\mathbf{p} = 2ikA\mathbf{L}\mathbf{e}, \quad (16)$$

where $\mathbf{p}$ is a column vector $(p_1, \dots, p_N)^T$ containing the acoustic pressure at each element, and $\mathbf{L}$ and $\mathbf{M}$ are $N \times N$ matrices with elements

$$\mathbf{L}_{l,m} = \int_{\Gamma_m \cap \Gamma_{\text{in}}} G_{x_l}(y) d\Gamma(y), \quad \mathbf{M}_{l,m} = \int_{\Gamma_m} \frac{\partial G_{x_l}(y)}{\partial n_y} d\Gamma(y), \quad (17)$$

where x_l is the collocation point in element Γ_l and $\mathbf{e} = (1, \dots, 1)^T$. The BEM approximation of the solution to integral equation (11) can thus be written

$$p^{(N)}(x) = \sum_{m=1}^N p_m \chi_{\Gamma_m}(x), \quad (18)$$

where χ_{Γ_m} is the characteristic function on Γ_m .

Remark 1. The numerical solution using this BEM formulation can be non unique at certain frequencies, as reported by Copley [4]. Several solutions have been proposed to overcome this problem [3, 15]. Fortunately, we did not encounter this problem for our setup in the frequency band we studied.

The design boundary Γ_d is divided into M conical elements, numbered from the throat to the mouth of the device. We choose design variable η to be a piecewise-constant function. In the discrete setting, a vector $\boldsymbol{\eta} = (\eta_1, \dots, \eta_M)^T$ contains the values of η in each strip. With this design representation, we solve initial value problems (14) and (15) exactly using integration to compute β_l , the distance from Γ_d^{ref} to the endpoint of Γ_l , for $l = 1, \dots, M$. We then define our discretized β to be continuous and piecewise linear and coinciding with the exact solution of the initial value problem at the nodes. Note that, for the type 2 parameterization, our discretized β coincide with the exact solution of (15) at all points. Finally, we define the vector $\boldsymbol{\beta} = (\beta_1, \dots, \beta_M)^T$ containing the perpendicular distances from the flare to Γ_d^{ref} at the nodes. The relationship between $\boldsymbol{\beta}$ and $\boldsymbol{\eta}$ for our two parameterization types is given by

$$\boldsymbol{\beta} = \tilde{\mathbf{L}}^{(1)}\boldsymbol{\eta} \text{ and } \boldsymbol{\beta} = \tilde{\mathbf{L}}^{(2)}\boldsymbol{\eta}, \quad (19)$$

where $\tilde{\mathbf{L}}^{(1)}$ and $\tilde{\mathbf{L}}^{(2)}$ are lower triangular matrices representing parameterization type 1 and 2 respectively.

Figure 3. Boundary deformations $\delta\varphi$ for $\delta\beta_l = 1$.

3.2. Objective Function Derivatives

Let $R^{(N)}$ denote the objective function (3) evaluated using the numerical solution $p^{(N)}$ defined in expression (18). The optimization algorithm requires gradients of objective function $R^{(N)}$ with respect to the design variable. To calculate $\nabla_{\boldsymbol{\eta}} R^{(N)}$, the gradient of $R^{(N)}$ with respect to design variable $\boldsymbol{\eta}$, we first evaluate $\nabla_{\boldsymbol{\beta}} R^{(N)}$, the gradient of $R^{(N)}$ with respect to $\boldsymbol{\beta}$; then, we obtain $\nabla_{\boldsymbol{\eta}} R^{(N)}$ using $\nabla_{\boldsymbol{\beta}} R^{(N)}$, the chain rule, and relation (19).

To calculate $\partial R^{(N)} / \partial \beta_l$, we will utilize expression (10) with design variation $\delta\beta^c$, marked gray in Figure 3. This design variation is in the direction n_r , normal to the reference shape, and can thus be written $\delta\varphi = n_r \delta\beta^c$. Moreover, for each element Γ_m , we define, with the aid of expression (12), an approximation of $\nabla p|_{\Gamma_m}$ as

$$D_m p^{(N)} = \tau_m \mathcal{I}(p^{(N)}, x_m, \tau_m) \quad (20)$$

where τ_m is the tangent vector of Γ_m . We calculate[†] the directional derivative $\delta R^{(N)}$ with respect to design variation $\delta\beta^c$ by replacing p and ∇p in expression (10) with the numerical approximations $p^{(N)}$ and $D_m p^{(N)}$. Thus,

$$\delta R^{(N)}(\delta\beta^c) = \frac{1}{2ikA^2|\Gamma_{\text{in}}|} \sum_{m=1}^M \left(\left[k^2 p_m^2 - D_m p^{(N)} \cdot D_m p^{(N)} \right] \int_{\Gamma_m} \delta\beta^c n_d \cdot n_r \, d\Gamma \right),$$

where n_d is normal to the current design boundary (Figure 3), and p_m denotes the constant acoustic pressure on Γ_m . For the current design variation, $\delta\beta^c$ is zero in all elements except Γ_l and Γ_{l+1} , where $\delta\beta^c$ is linear between 0 and $\delta\beta_l$, thus

$$\delta R^{(N)}(\delta\beta^c) = \frac{\delta\beta_l}{2ikA^2|\Gamma_{\text{in}}|} \left(\frac{h_l}{2} \left[k^2 p_l^2 - D_l \cdot D_l \right] + \frac{h_{l+1}}{2} \left[k^2 p_{l+1}^2 - D_{l+1} \cdot D_{l+1} \right] \right),$$

where h_l and h_{l+1} are width of element Γ_l and Γ_{l+1} in the direction of the reference plane, respectively. Thus, $\partial R / \partial \beta_l$ can be expressed as

$$\frac{\partial R^{(N)}}{\partial \beta_l} = \frac{1}{2ikA^2|\Gamma_{\text{in}}|} \left(\frac{h_l}{2} \left[k^2 p_l^2 - D_l \cdot D_l \right] + \frac{h_{l+1}}{2} \left[k^2 p_{l+1}^2 - D_{l+1} \cdot D_{l+1} \right] \right).$$

[†]Since expression (10) is derived for the equations before discretization, the procedure of this section will introduce discretization errors in the gradient directions, as further discussed in § 5.

By using the chain rule and expression (19), we obtain the expressions

$$\nabla_{\eta} R^{(N)} = \left[\tilde{\mathbf{L}}^{(1)} \right]^T \begin{pmatrix} \frac{\partial R}{\partial \beta_1} \\ \vdots \\ \frac{\partial R}{\partial \beta_M} \end{pmatrix} \text{ and } \nabla_{\eta} R^{(N)} = \left[\tilde{\mathbf{L}}^{(2)} \right]^T \begin{pmatrix} \frac{\partial R}{\partial \beta_1} \\ \vdots \\ \frac{\partial R}{\partial \beta_M} \end{pmatrix} \quad (21)$$

for parameterization type 1 and 2 respectively.

Remark 2. The back side of the horn moves when the last element of β changes. Thus, the pressure and its approximate gradient from all elements on the back side of the horn needs to be included in the evaluation of $\partial R / \partial \beta_M$.

3.3. Quadrature

In order to set up linear system (16), we need to compute matrices $\mathbf{M}$ and $\mathbf{L}$ with elements

$$\mathbf{L}_{l,m} = \int_{\Gamma_m} \frac{e^{-ik|x_l-y|}}{4\pi|x_l-y|} d\Gamma(y), \quad (22)$$

and

$$\mathbf{M}_{l,m} = \int_{\Gamma_m} \frac{e^{-ik|x_l-y|} (ik|x_l-y| + 1)}{4\pi|x_l-y|^2} \frac{(x_l-y) \cdot n_y}{|x_l-y|} d\Gamma(y), \quad (23)$$

where x_l is the collocation point in Γ_l . For the off-diagonal elements ($l \neq m$), the integrals are regular. We use the adaptive Simpson rule with tolerance 10^{-6} to evaluate these integrals.

For the diagonal elements, the collocation point is located on the element that we integrate over. We divide the element into two parts, one small strip $\tilde{\Gamma}_l$ around the collocation point and the rest of the strip $\Gamma_l \setminus \tilde{\Gamma}_l$. The integral over Γ_l equals the sum of the integrals over $\tilde{\Gamma}_l$ and $\Gamma_l \setminus \tilde{\Gamma}_l$. The integrals are regular on $\Gamma_l \setminus \tilde{\Gamma}_l$, and we apply the adaptive Simpson rule to evaluate integrals on this part. The strip $\tilde{\Gamma}_l$ is small enough to be well approximated by a flat surface. Assuming that $\tilde{\Gamma}_l$ is flat and performing a polar coordinate transform as illustrated in Figure 4, we find that

$$\int_{\tilde{\Gamma}_l} \frac{e^{ik|x-y|}}{4\pi|x-y|} dy = \int_0^{2\pi} \int_0^{R(\theta)} \frac{e^{-ikr}}{4\pi r} r d\theta dr = \frac{-1}{4ik\pi} \int_0^{2\pi} e^{-ikR(\theta)} d\theta + \frac{1}{2ik},$$

where the function $R(\theta)$ is the distance (in the direction corresponding to θ) from x_l to the boundary of $\tilde{\Gamma}_l$. Finally, the adaptive Simpson rule evaluates the remaining integral above over $[0, 2\pi]$.

In the gradient evaluation we evaluate $\partial p / \partial \tau_x$ by using expression (12). The pressure p is assumed to be constant at each element, thus we only need to evaluate integrals of the kinds

$$\int_{\Gamma_{\text{in}} \cap \Gamma_m} \frac{\partial G_x(y)}{\partial \tau_x} d\Gamma(y), \text{ and } \int_{\Gamma_m} \frac{\partial^2 G_x(y)}{\partial \tau_x \partial n_y} d\Gamma(y).$$

Fortunately, these integrals are regular—in the first $x \in \Gamma_d$ while $y \in \Gamma_{\text{in}}$, and the second is regular since τ_x is in the tangential direction. We evaluate also these integrals with the adaptive Simpson rule.

Figure 4. When the collocation point x_0 is located on the element we integrate over, we treat a small strip $\tilde{\Gamma}_0$ around this point separately.

4. Numerical Experiments and Results

4.1. Baseline spectra

Our first numerical experiment aims at validating our implementation of the boundary element method as well as to compute baseline spectra for two interface shapes. Our two reference shapes, illustrated on the right side of Figure 5, are a truncated waveguide (top) and a waveguide truncated with a funnel-shaped end part (bottom). The length of the funnel-shaped part is 0.5 m and the opening at the end has diameter 0.6 m. The diagram in Figure 5 shows their reflection spectra computed using our BEM implementation with 452 elements for both reference shapes as well as a finite element implementation employing piecewise quadratic elements with a total of 39,241 and 48,037 degrees of freedom for the truncated waveguide and the funnel-shaped flare, respectively. In all experiments in this section, we set the sound speed c to 345 m/s. For the finite element solution, we use the setup from Udawalpola and Berggren [17], solving the Helmholtz equation in cylindrical symmetry on a semidisk with radius 1.2 m using the lowest-order Engquist–Majda artificial boundary condition on the outer boundary to approximate the Sommerfeld condition. Comparing the spectra in Figure 5, we see that the results computed with the BEM and FEM are almost indistinguishable. The small differences between the spectra are due to different far-field conditions—the BEM implementation use the exact Sommerfeld radiation condition (2d), while the FEM implementation use a first order approximation—as well as discretization errors in both methods.

4.2. Optimization Results

For the optimization of the interfacial device, we minimize the amplitude of the reflected wave for frequencies throughout a pre-specified frequency band. We are primarily interested in seeing how the optimal shape depends on the chosen frequency band. The diameter of the waveguide is 10 cm and the length of the region that is allowed to move is 50 cm. This part

Figure 5. Reflection spectra for a conical horn (below) and a tube (above). The solid lines represent results obtained using BEM and dashed lines represent results with FEM.

of the device will henceforth be denoted the *flare*. For the optimization of the flare shape, we solve the following least-squares problem

$$\begin{aligned} \min_{\boldsymbol{\eta}} \quad & \sum_{f \in \mathbf{F}} |R_f(\boldsymbol{\eta})|^2 + \varepsilon \langle \boldsymbol{\eta}, \boldsymbol{\eta} \rangle \\ \text{s.t.} \quad & \boldsymbol{\eta} \geq 0 \end{aligned}$$

where $\mathbf{F}$ is a set of frequencies for which we want to minimize the reflections back to the waveguide. The last term provides, if needed, Tichonov regularization of the flare shape. We use Matlab's function `lsqnonlin`, with the large-scale option and using default termination conditions, to solve the above optimization problem. The algorithm in `lsqnonlin` requires a problem that is not underdetermined. In the numerical experiments presented in this section, we solve the problem with Tichonov parameter $\varepsilon = 10^{-8}$; thus the problem is practically unconstrained, but the presence of the Tichonov term ensures that the problem is never underdetermined.

We start by optimizing the device for frequency band 523–880 Hz, minimizing the reflection at the 10 frequencies $440 \cdot 2^{m/12}$ Hz, where $m = 3, 4, \dots, 12$, that is, all semitones from C5 to A5. For this case, we present results (only) from optimization with parameterization type 1 (convex flare shapes). The optimization converges in 65 iterations and the resulting coupling between the waveguide and free space is very efficient. More precisely, the portion of the wave that is reflected is less than 1.3 % for all frequencies in the range for which the flare is optimized. Figure 6 shows snapshots from the optimization. The upper row shows the initial shape (left), the shape after the first iteration (middle), and the shape after five iterations (right). These snapshots illustrate that in the beginning of the optimization process, the most important single feature that improves the performance is the opening of the device mouth. The middle row shows (from left to right) the design after 10, 20, and 30 iterations. The optimization

Figure 6. Optimization using convex flare shapes (type 1) for frequency band 523–880 Hz. The shapes show the evolution of the flare shape throughout the optimization process from initial tube (upper left) to final optimized horn (bottom right). The pictures show the device after (starting with the upper row from left to right) 0, 1, 5, 10, 20, 30, 40, 50, and 65 iterations.

routine continues opening the mouth of the device; however, we can also see some smaller features evolving. The bottom row illustrates the design after 40, 50, and 65 iterations. During the final part of the optimization, the trend of increasing mouth diameter stops; in fact, the mouth diameter decreases slightly during these last 25 iterations. At the same time, the throat of the device narrows to attain the same diameter as the waveguide for a longer distance. In essence, this shortens the length of the device and at the same time tunes the profile. The diameter of the resulting device stays smaller than 11 cm until a distance of 30 cm from the mouth and the final mouth diameter is 49 cm. As a comparison, the wavelengths corresponding to the frequencies the device is optimized for range from 39 cm (880 Hz) to 66 cm (523 Hz).

In our second experiment, we optimize the interfacial device using both parameterization types for three frequency bands. The first band covers the half octave from 622–880 Hz ($D^{\#}5$ to $A5$), the second starts a quarter octave lower and covers 523–880 Hz ($C5$ to $A5$). The third frequency band starts at the same frequency as the first frequency band, but ends a quarter octave higher, that is, the third frequency band is 622–1046 Hz ($D^{\#}5$ to $C6$). In the optimization, we minimize the reflection in all semitones (that is, frequencies of the kind $440 \cdot 2^{m/12}$ Hz, where m is an integer) in the frequency ranges above. Figures 7 and 8 depict

Figure 7. Convex flare shapes (type 1) optimized for different frequency bands: 523–880 Hz (left), 622–880 Hz (middle), and 622–1046 Hz (right).

Figure 8. Increasing flare shapes (type 2) optimized for different frequency bands: 523–880 Hz (left), 622–880 Hz (middle), and 622–1046 Hz (right).

Figure 9. Reflection spectra for the shapes in Figures 7 (left diagram) and 8 (right diagram). The solid line represents frequency band 622–880 Hz, the dashed line 523–880 Hz, and the dashed dotted line 622–1046 Hz.

the resulting shapes for type 1 (convex flare) and type 2 (increasing flare) parameterizations, respectively, and Figure 9 shows the reflection spectra for these devices. As can be seen from the reflection spectra, all horns have very small reflections throughout the frequency range for which they are optimized. Thus, all six shapes are essentially optimal in the range 622–880 Hz. This illustrates that the problem of finding an optimal interfacial device for this frequency range has many solutions.

Comparing the shapes corresponding to the different frequency ranges, there are some clearly visible similarities as well as differences between the devices. First, the devices optimized for

Figure 10. Convex flare shapes (type 1) optimized for different frequency bands: $440 \cdot (2^{m/12}, \dots, 2^{(m+6)/12})$, from left: $m = -6$, $m = -3$, $m = 0$, and $m = 3$.

Figure 11. Increasing flare shapes (type 2) optimized for different frequency bands: $440 \cdot (2^{m/12}, \dots, 2^{(m+6)/12})$, from left: $m = -6$, $m = -3$, $m = 0$, and $m = 3$.

the same frequency band, particularly those optimized for the wider (3/4 octave) frequency ranges, but using different parameterization types share general features and look quite similar. Examining instead the devices optimized using the same parameterization type but in different frequency bands, we see a trend in that the diameter stays narrow for a longer distance, but expands more rapidly near the end of the interfacial device the lower the frequency range becomes. Another perhaps interesting observation is that the mouth diameter is around 49 cm in three cases (low frequency range for type 1, and low and mid frequency ranges for type 2) and about 56 cm in the other three cases (mid and high frequency ranges for type 1, and high frequency range for type 2).

For acoustic horns, the required size of the device is typically dictated by the lowest frequency that is to be efficiently transmitted—the lower this frequency becomes the larger the horn gets. In our third set of experiments, we study low frequency interfacial devices. Here, we use half-octave-wide frequency bands starting at different notes. Figures 10 and 11 show the resulting shapes for type 1 and type 2 parameterizations, respectively, and Figure 12 the corresponding reflection spectra. The resulting shapes illustrate that the device mouth indeed gets larger as the frequencies that are to be transmitted decreases. This trend is clear for devices both with convex and increasing flare shapes. The mouth diameters for the convex flares in Figure 10 are (from left to right) 66 cm, 63 cm, 53 cm, and 48 cm; and the mouth diameters for the increasing flares in Figure 11 are (from left to right) 70 cm, 64 cm, 55 cm, and 54 cm. The shapes for the convex case looks rather well controlled and does not have any rapid changes, while for the increasing case the shapes gets wilder as lower frequencies are to be transmitted. Comparing

Figure 12. Reflection spectra for the shapes in Figures 10 (left diagram) and 11 (right diagram). The shapes are optimized for frequency bands $440 \cdot (2^{m/12}, \dots, 2^{(m+6)/12})$; solid lines correspond to $m = 6$, dashed lines correspond to $m = 3$, solid-crossed lines correspond to $m = 0$, and dash-dotted lines correspond to $m = -3$.

the reflection spectra in Figure 12, we see that using type 2 parameterization we obtain very efficient interfacial devices for all four frequency bands. When using type 1 parameterization we obtain very efficient interfacial devices for three of the four frequency bands. For the lowest frequency range, the convex flare is not optimal; the portion of the wave that is reflected is at most 5.3 %. In comparison, the increasing flare optimized for the same frequency range has a maximal reflection of 2.0 %. However, in this frequency range, both devices are superior to our reference shapes, which exhibit maximal reflections of 96 % (tube) and 79 % (funnel) respectively.

5. Discussion and Conclusions

To the best of our knowledge, this is the first time that the adjoint approach to gradient computations has been used in connection with the boundary element method in order to solve acoustic shape optimization problems. Note that gradient expression (10) is derived from the Helmholtz equations in differential form (as opposed to the boundary integral equation form). The Helmholtz equation is then discretized using the BEM, and formula (10) is evaluated for the numerical solution. In the literature, this approach is called differentiate-and-discretize or the “continuous” approach, as opposed to discretize-and-differentiate or the “discrete” approach. In the latter approach, the governing equation and the objective function are first discretized. The adjoint equation and the gradient expression are then derived for the discrete optimization problem. Since the processes of discretization and differentiation do not commute in general, the expressions obtained by the two approaches cannot be expected to coincide. Berggren [2] specifies the particular terms that differ when using the different approaches in the case of standard finite-element discretizations of elliptic problems. The question of which approach that should be preferred is somewhat controversial; Glowinski and He [8] and Gunzburger [11], among many others, discuss and offer perspectives on the issue. In our previous work on shape optimization for acoustic problems, which all used finite-element

discretizations, we consistently employed the “discrete” approach. The reason is that the discrete method in general can be expected to be more robust, since the expression that is used is then the true gradient of a function, something that cannot be guaranteed in the “continuous” case. However, the BEM discretization involves both an integral equation formulation and an involved adaptive quadrature scheme that is of essential importance to obtain sufficient accuracy of the diagonal and near diagonal matrix elements. Differentiating through this compound mapping seemed like an overly daunting task, which is why we chose instead to employ the “continuous” approach in the present case. Perhaps surprisingly—in the view of the complexity of the derivative calculations reported in section 3.2—the gradient accuracy turned out to be good enough to obtain a quite rapid convergence of the optimization algorithm. The particular structure of the problem considered here—the adjoint state is proportional to the state (expression (9))—could possibly be an explanation to the good performance of the “continuous” approach in this case.

A particular advantage with the boundary element method in connection with shape optimization is that the method, as opposed to finite-element or finite-difference methods, does not require remeshing or deformation of volume meshes. We exploited this freedom in the current study in order to let the mouth diameter of the device freely evolve in the optimization process. We conclude that there appears to be a natural limit that contains the optimal mouth diameter within reasonable bounds. Thus, an explicit bound on the mouth diameter seems not to be necessary in order to obtain a well-posed optimization problem.

Generally, the type 1 parameterization produced smoother shapes, which likely is connected with the fact that the design variable η then roughly corresponds to the axial *curvature* of the flare. Gradient-based optimization methods are local and tend thus to find local optima in the *neighborhood* of the initial shape. (Recall that there usually are multiple solutions to the problems considered here, particularly for narrow frequency bands.) “Neighborhoods” correspond to nearby values of the design variable (and not necessarily to shapes that are visually similar). Thus, for the type 1 parameterization, a design in the neighborhood of the initial shape is a shape with a similar value of the curvature, which means smooth shapes if the initial shape is smooth. The preference for smooth shapes when using the type 1 parameterization can be a particular advantage when optimizing in higher frequency bands, whereas it can be a liability for the lower frequencies; we obtained better transmission properties using the type 2 parameterization when optimizing in the difficult low-frequency region.

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