Course: 5DA001/Non-linear Optimization

Teachers in charge: Niclas Börlin, Klas Markström

Allowed aids: None.

Date: 2016-01-12

Time: 9.00–13.00

Max points: 100

Name: ____________________________________________

Civic registration number: __________________________

Code: 15

Note!

This examination will be graded anonymously. This sheet will be removed before the teacher receives the rest of the examination. The above code must therefore be on all other pages when you submit the examination to the examination supervisory staff.

Till skrivningsbevakaren: Avskilj detta försättsblad och stoppa i kuvert som skickas till Studentexpeditionen, Datavetenskap.
EXAM

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Remember:

• Start each problem on a new sheet of paper.

• Write your code and the problem number in the top right corner on each sheet.

• Mark the problems you have attempted to solve in the table to the right.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Attempted?</th>
<th>Points</th>
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  Sum

Grade
Some help for some problem

If \( v = [a, b, c]^T \), then
\[
\begin{bmatrix}
-b & -c \\
-a & 0 \\
0 & a
\end{bmatrix}^T v = 0.
\]

\[
M_1 = \begin{bmatrix}
2 & 2 \\
2 & 3
\end{bmatrix}, \quad |M_1 - \lambda I| = \left( \lambda - \frac{5 + \sqrt{17}}{2} \right) \left( \lambda - \frac{5 - \sqrt{17}}{2} \right).
\]

\[
M_2 = \begin{bmatrix}
-2 & 2 \\
2 & 3
\end{bmatrix}, \quad |M_2 - \lambda I| = \left( \lambda - \frac{1 + \sqrt{41}}{2} \right) \left( \lambda - \frac{1 - \sqrt{41}}{2} \right).
\]

\[
M_3 = \begin{bmatrix}
2 & -2 \\
-2 & 3
\end{bmatrix}, \quad |M_3 - \lambda I| = \left( \lambda - \frac{5 + \sqrt{17}}{2} \right) \left( \lambda - \frac{5 - \sqrt{17}}{2} \right).
\]

\[
M_4 = \begin{bmatrix}
2 & 2 \\
2 & -3
\end{bmatrix}, \quad |M_4 - \lambda I| = \left( \lambda - \frac{-1 + \sqrt{41}}{2} \right) \left( \lambda - \frac{-1 - \sqrt{41}}{2} \right).
\]

Problem 1: (13p)

Study the twice continuously differentiable function \( f(x) \in \mathbb{R}, x \in \mathbb{R}^n \).

1.1) Define the terms local minimizer, global minimizer and strict and non-strict minimizer (four definitions in total) for \( f \). (Hint: The definitions do not assume that \( f \) is differentiable.)

1.2) Under what circumstances are local minimizers unique and non-unique, respectively?

1.3) Suggest a function without a local minimizer.

Problem 2: (30p)

2.1) Consider the function
\[
f(x) = x_1^2 - 2x_1 + x_2^2 - x_3^2 + 4x_3.
\]

Find all stationary points to the problem
\[
\min_x f(x).
\]

For each stationary point, state whether it is a local minimizer, local maximizer, or a saddle point. Motivate!

2.2) Study the problem
\[
\min_x \quad f(x) \\
\text{s.t.} \quad x_1 - x_2 + 2x_3 = 2.
\]

Determine which, if any, of the points \( x = (1, 0, 2)^T, x = (2.5, -1.5, -1)^T, \) and \( x = (1.5, -2.5, -1)^T \) that is a local minimizer to the constrained problem. Motivate!
Problem 3:  (19p)

3.1) Describe the steps in the general Branch and Bound-algorithm.

3.2) Which of the steps in the general algorithm needs to be specified in order to get an algorithm for a specific problem?

Problem 4:  (16p)

4.1) Consider the problem

\[ \min_x \frac{1}{2} \|Ax - b\|^2, \]  

where \( A \in \mathbb{R}^{m \times n}, m \geq n \) has full rank, and \( b \in \mathbb{R}^m \).

Formulate problem (1) as a non-linear least squares problem and state the residual function \( r(x) \) and Jacobian function \( J(x) \).

4.2) Given \( x_0 = 0 \in \mathbb{R}^n \) as the starting approximation, give the expression for \( x_1 \) using the Gauss-Newton method without linesearch.

4.3) On average, how many iterations are required for the Gauss-Newton method to converge on problem (1).

Problem 5:  (14p)

A discrete problem with variables \( x = (x_1, x_2, \ldots, x_n) \), and objective function \( f(x) \) and a neighbourhood \( N(x) \) are given.

5.1) Give a complete description of the first improvement version of a local search algorithm for this problem.

5.2) What is required to make sure that this algorithm can find the global optimum from any initial point?

Problem 6:  (8p)

6.1) For a non-linear least squares problem, the Gauss-Newton search direction is computed as the solution of the Newton equation with the Hessian approximated by

\[ H(x_k) = J(x_k)^T J(x_k). \]

If \( H(x_k) \) is singular, the Newton equation does not have a solution. In that situation, what does that tell us about \( J(x_k) \)?

6.2) If \( H(x_k) \) is singular for some \( k \), the Gauss-Newton method will fail. Suggest how the Gauss-Newton method could be modified to handle such situations and still be able to converge toward a minimizer \( x^* \) assuming \( H(x^*) \) has full rank.