Assignment 3

Matrix Computations and Applications
1 Theoretical problems

The numbers of the exercises below refer to the 4th edition of the book. Learn how to solve these problems using any means necessary. We recommend that you first give it a serious attempt on your own before seeking hints. This part of the assignment will be assessed orally as follows: You will be asked to solve a few of the problems listed below on a whiteboard without any aids. The problems will be chosen at random. Time and place will be announced on the course web.

§4.1: 4, 6, 9, 10, 13, 14, 16, 17, 18, 23, 25.
§4.2: 1, 3, 4, 5, 9, 11, 13, 16, 17, 19, 20, 21, 26, 30.
§4.4: 2, 6, 10, 16, 20, 30, 31, 33.
§6.1: 2, 3, 4, 8, 9, 26, 32.
§6.2: 1, 2, 3, 6, 7, 14, 15, 19, 21, 23, 25, 26, 30, 32, 33, 37.
§6.3: 1, 2, 3, 8, 10, 11, 12, 14, 15, 18, 26.
§6.4: 1, 8, 11, 12, 15, 18.
§6.7: 4, 5, 8, 13, 14, 17.

2 Practical problems

This part of the assignment should be solved individually. You may discuss problems with others in general terms, but the end product must be 100% your own. For example, it is strictly prohibited to collaborate and to copy other sources. Hand in a printed copy of your source code including a path to the digital version. Also hand in a brief report describing your results. You will primarily be judged by the quality of the source code and secondarily by the presentation of your results.

2.1 Differential equations

In this exercise, we will study three different discretizations of the differential equation

\[ y'' = -y. \]  

(1)

Complete the following exercises.

1. Write (1) as a first-order system

\[ \frac{du}{dt} = Au, \quad \text{where} \quad u = \begin{bmatrix} y \\ y' \end{bmatrix} \]  

(2)

with initial conditions \( y(0) = 1 \) and \( y'(0) = 0 \). Find the exact solution to the first-order system (2) by diagonalizing the \( 2 \times 2 \) matrix \( A \). Evaluate the exact solution at 33 uniformly distributed points in the interval \([0, 2\pi]\) (including the endpoints). Display the solution in the plane. The points should lie on a circle with unit radius. (Hint: make sure the axis are scaled equally, or else you will see an ellipse instead of a circle.)

2. Discretize (2) in three different ways:

   (a) By \textit{forward} differences in both components.
   
   (b) By a \textit{forward} difference in the first component and a \textit{backward} difference in the second component.
   
   (c) By \textit{backward} differences in both components.
Express these schemes in the form of
\[ u_{n+1} = Bu_n, \]
where \( B \) is a \( 2 \times 2 \) matrix (different for each scheme). Apply the three schemes for 32 time steps starting at the initial vector \( u_0 = (1, 0)^T \) and with time step \( \Delta t = 2\pi/32 \). Compare with the exact solution. Explain the three different behaviours (for general time steps \( \Delta t \)) by analyzing the eigenvalues of the time-stepping matrices (i.e., the three \( B \)-matrices above).

2.2 Ranking of web pages

1. Implement the power method (see page 487 in Strang) as a MATLAB function

```matlab
function x = powermethod(matvec, normalize, x0, tol)
```

where
- \( \text{matvec} \) is a \textit{function} that takes a vector as input, pre-multiplies it with a matrix, and returns the resulting vector.
- \( \text{normalize} \) is a \textit{function} that takes a vector as input and returns a normalized version of that vector.
- \( x0 \) is an initial starting vector.
- \( \text{tol} \) is a tolerance parameter used in the convergence criterion.

2. Implement the PageRank algorithm based on your \texttt{powermethod} function by defining appropriate functions \texttt{matvec} and \texttt{normalize}.

- Denote the raw web matrix by \( P \), where \( p_{ij} = 1 \) if there is a link from \( j \) to \( i \).
- Handle dangling nodes by replacing the corresponding zero columns by \( e/n \). Denote the resulting matrix by \( \bar{P} \).
- Force irreducibility by using the convex combination

\[ \tilde{P} := \alpha \bar{P} + (1 - \alpha)ee^T, \quad 0 < \alpha < 1 \]

as input to the power method.
- (Important) Exploit the sparsity of \( \tilde{P} \) in the implementation of \texttt{matvec}.

3. Apply your PageRank implementation to the web in the \texttt{teknat09.mat} file that you can find on the course web. With \( \alpha = 0.85 \), which are the ten most highly ranked pages and what are their PageRanks? Show convergence plots (with logarithmic scale on the \( y \)-axis) for \( \alpha \in \{0.1, 0.85, 0.99\} \).