The purpose of this project is to develop and test a MATLAB function f.m which can be used to calculate the function \( g(x) = \sin(x) \) for \( x \in [-\pi, \pi] \).

Your function must be odd, i.e.

\[
f(x) = -f(-x)
\]

and certain special values must coincide exactly with the definition of the sine function, see Table 1.

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-\pi)</th>
<th>(-\frac{3\pi}{4})</th>
<th>(-\frac{\pi}{2})</th>
<th>(-\frac{\pi}{4})</th>
<th>0</th>
<th>(\frac{\pi}{4})</th>
<th>(\frac{\pi}{2})</th>
<th>(\frac{3\pi}{4})</th>
<th>(\pi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>0</td>
<td>(-\sqrt{2}/2)</td>
<td>(-1)</td>
<td>(-\sqrt{2}/2)</td>
<td>0</td>
<td>(\sqrt{2}/2)</td>
<td>1</td>
<td>(\sqrt{2}/2)</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 1: Your implementation must return these values.

You are allowed to use the MATLAB function “sqrt” as well as MATLAB’s constant “pi”.

Below is a list of theoretical questions which are designed to guide you towards a suitable implementation. You should hand in the following material.

1. Your solution of the theoretical questions.
2. Your own implementation of the sine function.
3. Documentation describing the algorithm and the call sequence.
4. A comparison between your own implementation and MATLAB’s built-in function.

You should hand in your solution by noon on

Monday, October 3rd, 2011.
You are allowed to work together in groups of 2 or 3 people. One report per group will suffice, but remember to state the name of all group members.

These are the theoretical questions. Let \( g : \mathbb{R} \to \mathbb{R} \) be given by \( g(x) = \sin(x) \). Take note of the fact that the interval \([0, \frac{\pi}{2}]\) is broken into three parts which are studied individually.

1. Show that the Taylor polynomial of order \( 2k - 1 \) for \( g \) at \( x = 0 \) is
   \[
   T_{2k-1}(x) = \sum_{j=0}^{k-1} \frac{(-1)^j}{(2j+1)!} x^{2j+1}.
   \]

2. Show that for all \( x \in \mathbb{R} \) there exists \( \xi \in \mathbb{R} \), such that
   \[
   g(x) - T_{2k-1}(x) = \frac{1}{(2k)!} g^{(2k)}(\xi) x^{2k}
   \]
   (Hint: This is an application of Taylor’s formula.)

3. Let \( x \in [0, \frac{\pi}{8}] \). Show that the absolute error satisfies
   \[
   |T_{2k-1}(x) - g(x)| \leq \frac{1}{(2k)!} |x|^{2k} \leq \frac{1}{(2k)!} \left( \frac{\pi}{8} \right)^{2k}.
   \]

4. Let \( x \in (0, \frac{1}{8}\pi] \). Show that the relative error satisfies
   \[
   \frac{|T_{2k-1}(x) - g(x)|}{|g(x)|} \leq \frac{2}{(2k)!} |x|^{2k-1} \leq \frac{2}{(2k)!} \left( \frac{\pi}{8} \right)^{2k-1-1}
   \]
   (Hint: Show that \( 0 < \frac{1}{8}x < \sin(x) \) for \( x \in (0, \frac{1}{8}\pi] \).

5. Find the smallest \( k \) which will ensure that the relative error satisfies
   \[
   \frac{|T_{2k-1}(x) - g(x)|}{|g(x)|} \leq 2^{-53}
   \]
   for all \( x \in (0, \frac{1}{8}\pi] \).
   (Comment: Trial and error is not a bad way to go about this problem.)

6. Show that the \( k \)th order Taylor polynomial for \( g \) at the point \( x = \frac{\pi}{4} \) is given by
   \[
   U_k(x) = \sum_{j=0}^{k} (-1)^{\lfloor j/2 \rfloor} \sqrt{\frac{1}{2}} \left( \frac{x - \frac{\pi}{4}}{j!} \right)^j
   \]
   where \( \lfloor y \rfloor \) denotes the largest integer smaller than \( y \), ex. \( \lfloor \frac{3}{2} \rfloor = 1 \).

7. Show that there exists a \( \xi \in \mathbb{R} \) such that
   \[
   g(x) - U_k(x) = \frac{1}{(k+1)!} g^{(k+1)}(\xi) \left( x - \frac{\pi}{4} \right)^{k+1}
   \]

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8. Let \( x \in \left[ \frac{\pi}{8}, \frac{3\pi}{8} \right] \). Show that the absolute error satisfies

\[
|U_k(x) - g(x)| \leq \frac{1}{(k+1)!} \left| x - \frac{\pi}{4} \right|^{k+1} \leq \frac{1}{(k+1)!} \left( \frac{\pi}{8} \right)^{k+1}.
\]

9. Let \( x \in \left[ \frac{\pi}{8}, \frac{3\pi}{8} \right] \). Show that the relative error satisfies

\[
\left| \frac{U_k(x) - g(x)}{g(x)} \right| \leq \frac{4}{(k+1)!} \left| x - \frac{\pi}{4} \right|^{k+1} \leq \frac{4}{(k+1)!} \left( \frac{\pi}{8} \right)^{k+1}
\]

(Hint: Show that \( 1/4 \leq \sin(x) \) for the interval in question.)

10. Find the smallest integer \( k \) which will ensure that the relative error satisfies

\[
\left| \frac{U_k(x) - g(x)}{g(x)} \right| \leq 2^{-53}
\]

for all \( x \in \left[ \frac{\pi}{8}, \frac{3\pi}{8} \right] \).

11. Show that the Taylor polynomial of order \( 2k \) for \( g \) at the point \( x_0 = \frac{\pi}{2} \) is

\[
V_{2k}(x) = \sum_{j=0}^{k} (-1)^j \frac{(x - \frac{\pi}{2})^{2j}}{(2j)!}
\]

12. Show that there exists a \( \xi \in \mathbb{R} \) such that

\[
g(x) - V_{2k}(x) = \frac{1}{(2k+1)!} g^{(2k+1)}(\xi) \left( x - \frac{\pi}{2} \right)^{(2k+1)}
\]

13. Let \( x \in \left[ \frac{3\pi}{8}, \frac{5\pi}{8} \right] \). Show that the absolute error satisfies

\[
|V_{2k}(x) - g(x)| \leq \frac{1}{(2k+1)!} \left| x - \frac{\pi}{2} \right|^{(2k+1)} \leq \frac{1}{(2k+1)!} \left( \frac{\pi}{8} \right)^{(2k+1)}
\]

14. Show that the relative error satisfies

\[
\left| \frac{V_{2k}(x) - g(x)}{g(x)} \right| \leq \frac{2}{(2k+1)!} \left| x - \frac{\pi}{2} \right|^{(2k+1)} \leq \frac{2}{(2k+1)!} \left( \frac{\pi}{8} \right)^{(2k+1)}
\]

(Hint: You need a suitable lower bound on \( g \).)

15. Finally, find the smallest integer \( k \) such that the relative error satisfies

\[
\left| \frac{V_{2k}(x) - g(x)}{g(x)} \right| \leq 2^{-53}.
\]

In short, by using three different polynomials it is possible to obtain an accurate approximation of \( g \) on the interval \([0, \pi/2]\). You should handle the rest of the interval \([-\pi, \pi]\) by exploiting the symmetry of the sine function around the lines \( x = \frac{\pi}{2} \) and \( x = 0 \). You will need several conditional statements.