1 Instructions

You need to hand in the following material

1. Your answers to questions in the next section.
2. Your implementation of all MATLAB functions.
3. Documentation describing the purpose, call sequence and underlying algorithm of every MATLAB function which you have implemented.

You are free to use any piece of code which I have made available on the class website.

You should hand in your solution by noon on

Tuesday, January 15rd, 2013.

You are allowed to work together in groups of 2 or 3 people. One report per group will suffice, but remember to state the name of all group members.

The drop box is located immediately to the right of the entrance to the D-corridor at “Datavetenskap”.

2 Introduction

The purpose of this project is to develop the skills necessary to construct a firing table for a piece of artillery. This involves making certain simplifying assumptions, obtaining the correct differential equations, calculating the trajectories of the shells and determining certain parameters characterizing these trajectories.
2.1 Assumptions

For the sake of simplicity we will assume that the Earth is flat, that the gravitational field is uniform, and that the viscosity of the air is constant. We will ignore the rotation of the Earth, the spin of the projectile and we will not include any wind in our simulations. As a result of these assumptions the projectiles will be moving in a two dimensional plane.

2.2 The differential equations

Let 
\[ r(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \quad \text{and} \quad v(t) = \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} \]
denote the position and the velocity of the projectile at time \( t \). By Newton’s 2nd law the differential equation for the shell is
\[
\frac{d}{dt} v(t) = \dot{v}(t) = g - \frac{k}{m} \|v\|_2 v
\]
(1)

where \( g \) is the gravitational constant, \( k \) is a positive constant which depends on the shape of the shell and \( m \) is the mass of the shell. This can be written as a system of first order differential equations,
\[
z'(t) = f(z(t))
\]
where
\[
z(t) = \begin{bmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}, \quad \text{and} \quad f(z) = \begin{bmatrix} z_3 \\ z_4 \\ -\frac{k}{m} \left( \sqrt{z_3^2 + z_4^2} \right) z_3 \\ -g - \frac{k}{m} \left( \sqrt{z_3^2 + z_4^2} \right) z_4 \end{bmatrix},
\]
(2)

and \( g \) is the size of the local gravitational acceleration. At the surface of the Earth \( g \approx 9.82 \text{m/s}^2 \).

2.3 The initial condition

The shells are fired with an initial velocity \( v_0 \) and an angle \( \theta \), where \( \theta_0 = 90^\circ \) is straight up and \( \theta_0 = 0^\circ \) is parallel to the Earth. For the sake of simplicity we will ignore the barrel of the gun. Hence, the initial condition for the shell is
\[
z_0 = \begin{bmatrix} 0 \\ 0 \\ v_0 \cos(\theta_0) \\ v_0 \sin(\theta_0) \end{bmatrix}.
\]
2.4 The explicit Euler method

The explicit Euler method for the above initial value problem is merely the iteration
\[ w_0 = z_0, \quad w_{j+1} = w_j + hf(w_j), \quad j = 0, 1, 2, \ldots, \]
where \( h > 0 \) is the temporal step size.

2.5 How to determine the range

The range of the gun is the distance between gun and the point where the shell impacts the Earth. Similarly, the flight time \( \tau \) is the length of the time between firing the gun and the impact of the shell. Since we are only approximating the trajectory at a sequence of discrete points, specifically \( t_j = jh \), we must expect to find a \( j \), such that
\[ y(t_j) \approx w_2(t_j) > 0 \quad \text{and} \quad y(t_{j+1}) \approx w_2(t_{j+1}) < 0. \]

Obviously, if \( h \) is sufficiently small, then \( w_2(t_j) \) is a good approximation of \( y(t_j) \) and so
\[ y(t_j) > 0 \quad \text{and} \quad y(t_{j+1}) < 0 \]
which implies that the shell will impact the ground at some time between \( t_j \) and \( t_{j+1} \), i.e.
\[ \tau \in (t_j, t_{j+1}). \]
Can we be more specific? What we can do is to determine \( \sigma = \bar{\sigma} \in (0, 1) \), such that the second component of the vector \( w_{j+1}(\sigma) \) given by
\[ w_{j+1}(\sigma) = w_j + \sigma hf(w_j) \]
is exactly zero, and then use
\[ \bar{\tau} = t_j + \bar{\sigma}h \]
as an approximation for the true flight time \( \tau \). By the same token, the range can be approximated with the first component of \( w_{j+1}(\bar{\sigma}) \).

Obviously, the range is a function of the initial angle \( \theta_0 \) and the muzzle velocity \( v_0 \). Typically, the muzzle velocity of the gun is fixed and we can determine the maximum range of the gun by varying \( \theta_0 \) in an intelligent manner.

3 Questions

We will use the following data
\[ g = 9.82 \text{m/s}^2, \quad k = 7.035 \times 10^{-4} \text{kg/m}, \quad m = 9.1 \text{kg}, \quad v_0 = 820 \text{m/s} \]
which corresponds loosely to the famous German Flak 36 8.8cm gun from WWII. It was designed as an anti-aircraft gun, but was used extensively in an anti-tank capacity. It could be fired at angles between \(-3^\circ\) and \(90^\circ\).
1. Write a MATLAB routine which executes the explicit Euler method for a general initial value problem

\[ z'(t) = f(z(t)), \quad z(a) = z_0 \]

Call sequence

\[ [t \ w]=rk1(f,z0,a,b,N), \]

where \( f \) is a handler to a function, \( z0 \) is a column vector describing the initial condition, \([a, b]\) is the relevant time interval and \( N \) is the number of subintervals. The stepsize \( h \) will be \( h = (b - a)/N \). The subroutine must automatically detect the number of rows of \( z0 \), using, say, the \texttt{size} command, and build a two dimension array \( w \) with \( N+1 \) columns, so that the \((j+1)\)st column of \( w \) is an approximation of \( z(t_j) \). It is convenient to transpose \( w \) immediately before exiting the function, so that you get a tall, rather than a wide matrix. Finally, the vector \( t \) should contain the actual time steps, i.e.

\[ t(j) = a + (j - 1)h, \quad j = 1, 2, \ldots, N + 1 \]

2. Verify that the differential equation (1) for the shell can be written as a system of first order differential equations, i.e.

\[ \frac{d}{dt}z(t) = f(z(t)) \]

where \( f \) is given by equation (2).

3. Write a MATLAB routine \texttt{shell.m} which returns \( f(z) \) for the specific function describing the flight of a shell through the atmosphere.

Call sequence

\[ y = \text{shell}(z). \]

It is not convenient to pass the values of \( g \), \( k \), and \( m \) to the subroutine. Declare them as global variables, keyword \texttt{global}.

4. Write a MATLAB routine \texttt{range.m} which fires your gun and determines the range and flight time. Call sequence

\[ [\text{xmax} \ \tau \ \text{flag}] = \text{range}(\text{theta},v0,T,N), \]

where \( \text{theta} \) is the firing angle, \( v0 \) is the muzzle velocity, \( T \) is the number of seconds that you are going to simulate and \( N \) is the number of time steps. The routine must detect if the projectile has impacted with the ground during the simulated time interval (\texttt{flag} = 0) and issue an appropriate error message (\texttt{flag} = -1) if the shell is still in flight after \( T \) seconds. The \texttt{find} and \texttt{isempty} commands should prove very useful for this purpose. If the range can be determined, then the function should return the range as \texttt{xmax} and the flight time as \texttt{tau}.
5. It is clear that the computed range of your gun is a function of the stepsize $h$ which you are using, i.e.

$$x_{\text{max}} = x_{\text{max}}(h)$$

Verify numerically that the error for $x_{\text{max}}$ is $O(h)$ if you use the explicit Euler method to compute the trajectory. As usual, this can be done using stepsizes $h$, $2h$ and $4h$ to determine three different approximations of the target, i.e. the true value of the range. Once the order has been determined, then error estimates are readily available in the usual manner.

6. Construct a partial firing table for the Flak 36 cannon. For each angle

$$\theta_0 = 5, 10, 15, 20, 25, \ldots, 85$$

compute the range and the flight time using $v_0 = 820 \text{ m/s}$. Moreover, the range must be determined with an absolute error which is less than 10 meters. This may seem like a large error, but remember that our model is only a crude approximation of the physical reality. Normally, we are interested in the relative error, but since the kill radius of the shell is fixed it makes sense to focus on the absolute error.

7. Find the maximum range of the gun by varying the angle $\theta_0$ in an intelligent manner. **Hint:** It will be greater than 17.5 km. This figure is greater than the actual maximum range of the Flak 36 cannon. The problem hinges primarily on the fact that the assumption of a uniform atmosphere is incorrect.