1 Gaussian elimination with partial pivoting

This note is “interactive” in the sense that you stand to gain the most, if you manually verify every equality by doing the relevant matrix multiplications.

1.1 An example

Let \( A \) be matrix given by

\[
A = \begin{bmatrix}
1 & 0 & 1 \\
2 & 5 & -2 \\
3 & 6 & 9 \\
\end{bmatrix}.
\]

Focus on the first column of \( A_1 = A \). We decide to interchange row 3 and row 1, because row 3 contains the element which has the largest absolute value. Notice that this interchange can be carried out by multiplying with the matrix

\[
P_1 = \begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0 \\
\end{bmatrix}.
\]

Therefore

\[
\tilde{A}_1 = P_1 A_1 = \begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0 \\
\end{bmatrix} \begin{bmatrix}
1 & 0 & 1 \\
2 & 5 & -2 \\
3 & 6 & 9 \\
\end{bmatrix} = \begin{bmatrix}
3 & 6 & 9 \\
2 & 5 & -2 \\
1 & 0 & 1 \\
\end{bmatrix}.
\]

Now you are ready to clear the first column of \( \tilde{A}_1 \). Define

\[
M_1 = \begin{bmatrix}
1 & 0 & 0 \\
-\frac{2}{3} & 1 & 0 \\
-\frac{1}{3} & 0 & 1 \\
\end{bmatrix}.
\]

Then

\[
A_2 = M_1 \tilde{A}_1 = \begin{bmatrix}
1 & 0 & 0 \\
-\frac{2}{3} & 1 & 0 \\
-\frac{1}{3} & 0 & 1 \\
\end{bmatrix} \begin{bmatrix}
3 & 6 & 9 \\
2 & 5 & -2 \\
1 & 0 & 1 \\
\end{bmatrix} = \begin{bmatrix}
3 & 6 & 9 \\
0 & 1 & -8 \\
0 & -2 & -2 \\
\end{bmatrix}.
\]
Then focus on the second column of \( A_2 \). We interchange row 2 and row 3, because row 3 contains the element which has the largest absolute value. Notice that this interchange can be carried out by multiplying with the matrix
\[
P_2 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix}.
\]
Therefore
\[
\tilde{A}_2 = P_2 A_2 = \begin{bmatrix}
1 & 0 & 3 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix} \begin{bmatrix}
3 & 6 & 9 \\
0 & 1 & -8 \\
0 & -2 & -2
\end{bmatrix} = \begin{bmatrix}
3 & 6 & 9 \\
0 & -2 & -2 \\
0 & 1 & -8
\end{bmatrix}.
\]
Now, you are ready to clear the second column of \( \tilde{A}_2 \). Define
\[
M_2 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & \frac{1}{2} & 1
\end{bmatrix}.
\]
Then
\[
A_3 = M_2 \tilde{A}_2 = \begin{bmatrix}
1 & 0 & 3 \\
0 & 0 & 1 \\
0 & \frac{1}{2} & 1
\end{bmatrix} \begin{bmatrix}
3 & 6 & 9 \\
0 & -2 & -2 \\
0 & 1 & -8
\end{bmatrix} = \begin{bmatrix}
3 & 6 & 9 \\
0 & 0 & -9 \\
0 & 0 & -9
\end{bmatrix}.
\]
We have now arrived at an upper triangular matrix and therefore we define
\[
U = A_3 = \begin{bmatrix}
3 & 6 & 9 \\
0 & -2 & -2 \\
0 & 0 & -9
\end{bmatrix}.
\]
What is the relationship between all the matrices which we have named until this point? By construction we have
\[
U = A_3 = M_2 \tilde{A}_2 = M_2 P_2 A_2 = M_2 P_2 M_1 \tilde{A}_1 = M_2 P_2 M_1 P_1 A_1 = M_2 P_2 M_1 P_1 A.
\]
It follows that
\[
A = P_1^{-1} M_1^{-1} P_2^{-1} M_2^{-1} U.
\]
At this point it is prudent to calculate all these matrices. You should verify, that
\[
P_1^{-1} = P_1 = \begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}, \quad P_2^{-1} = P_2 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix}.
\]
This is not a surprise, because interchanging any two rows can be undone by interchanging the same two rows, one more time. In addition, you should verify that
\[
M_1^{-1} = \begin{bmatrix}
1 & 0 & 0 \\
\frac{1}{2} & 1 & 0 \\
\frac{1}{2} & 0 & 1
\end{bmatrix}, \quad M_2^{-1} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & -\frac{1}{2} & 1
\end{bmatrix}.
\]
Notice that the only difference between say $M_1$ and $M_1^{-1}$ is that the subdiagonal entries have changed sign. Why is this not a surprise? How can you undo adding a multiple of, say, row 1 to row 3? By subtracting the same multiple of row 1 from the new row 3.

Now, we reach a critical stage. We want a relation of the form

$$PA = LU,$$

where $L$ is lower triangular, $U$ is upper triangular and $P$ is not too complicated. Define

$$P = P_2P_1 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

and let us see what we can achieve! We have

$$P_2P_1A = P_2P_1(P_1^{-1}M_1^{-1}P_2^{-1}M_2^{-1}U) = P_2M_1^{-1}P_2^{-1}M_2^{-1}U.$$  

The good news is that we got rid of $P_1$ and $P_1^{-1}$. Now, verify manually

$$P_2M_1^{-1}P_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Again, this is not a surprise. We know that the effect of multiplying with $P_2$ from the left is to interchange rows 2 and 3. Similarly, the effect of multiplying with $P_2^{-1} = P_2$ from the right is to interchange columns 2 and 3. Therefore

$$P_2M_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

and

$$P_2M_1^{-1}P_2^{-1} = (P_2M_1^{-1})P_2 = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$  

In short, we have

$$PA = (P_2M_1^{-1}P_2^{-1})M_2^{-1}U = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} U.$$  

Now, define $L$ by

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$  


and verify that
\[
L = \begin{bmatrix}
1 & 0 & 0 \\
\frac{1}{3} & 1 & 0 \\
\frac{1}{3} & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & -\frac{1}{2} & 1
\end{bmatrix}
= (P_2M_1^{-1}P_2^{-1})M_2^{-1}.
\]

At this point you must verify, that
\[
PA = \begin{bmatrix}
3 & 6 & 9 \\
1 & 0 & 1 \\
2 & 5 & -2
\end{bmatrix}
= LU.
\]

Solving $Ax = f$ is now straightforward, because $PAx = LUx = Pf$ and the problem breaks down into the question of solving first $Ly = Pf$, and then $Ux = y$.

1.2 Why do the row interchanges at all?

Why do we bother with the row interchanges if they are not “strictly” necessary? This can be explained on many different levels. At this point, the best thing you can do is the following problem. It should convince you of the very real need to pivot.

**Problem 1** Consider the linear system
\[
\begin{bmatrix}
0.0002 & -0.00031 & 0.0017 \\
5 & -7 & 6 \\
8 & 6 & 3
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= \begin{bmatrix}
0.00609 \\
7 \\
2
\end{bmatrix}
\]

a. Verify that the exact solution is $(x_1, x_2, x_3) = (-2, 1, 4)$

b. Solve the linear system using Gaussian elimination with no pivoting and by rounding the result of each addition, subtraction, multiplication and division to 3 significant figures. Let $(\hat{x}_1, \hat{x}_2, \hat{x}_3)$ denote your solution.

c. Compare your approximate solution to the exact solution, by computing the relative errors
\[
\frac{|x_i - \hat{x}_i|}{|x_i|}, \quad i = 1, 2, 3
\]

(Do not be surprised if the relative error reaches 25 percent.)

d. Now solve the linear system using Gaussian elimination with pivoting and by rounding the result of each addition, subtraction, multiplication and division to 3 significant figures. Let $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ denote your solution.

e. Compare your approximate solution to the exact solution, by computing the relative errors
\[
\frac{|x_i - \tilde{x}_i|}{|x_i|}, \quad i = 1, 2, 3
\]

(The relative error should drop to less than 1 percent.)