1 Numerical solution of ODE

1.1 The basic idea

Let \( f : \mathbb{R} \to \mathbb{R} \) be a smooth function and consider the initial value problem

\[
y'(t) = f(y(t)), \quad t \in \mathbb{R} \\
y(0) = y_0, \quad y_0 \in \mathbb{R}
\]

Frequently, it is either extremely difficult to obtain a formula for the exact solution \( y \) or the formula itself may be very difficult to evaluate, so we settle for a far more modest goal. Specifically, we want to compute an approximation of \( y \) at a set of equally spaced points in a finite interval \([0, T]\).

To this end, we introduce the concept of a grid. Let \( h > 0 \) be given. Then we define the grid \( \Sigma_h \) as the set

\[
\Sigma_h = \{ jh : j \in \mathbb{N} \}
\]

A grid function \( v \) is merely a function defined on the grid, i.e. \( v : \Sigma_h \to \mathbb{R} \).

There exist a variety of techniques to generate grid functions which can be used to approximate the restriction of the exact solution \( y \) to the grid, i.e. the grid function \( y|_{\Sigma_h} : \Sigma_h \to \mathbb{R} \). The simplest possible method is the forward or explicit Euler method given by the following iteration

\[
v_0 = y_0 \\
v_{j+1} = v_j + hf(v_j), \quad j = 0, 1, 2, \ldots
\]

In many cases it is theoretically possible to bound the global error, but the theoretical bound is often quite useless. We require a more flexible approach.

1.2 Richardson extrapolation

Let \( T > 0 \) and let \( N > 0 \) be a positive integer. Let \( \bar{h} = T/N \) and let

\[
\Lambda = \{ j\bar{h} : j = 0, 1, 2, \ldots, N \} \subset [0, T]
\]
be a set of equally spaced points in the finite interval $[0, T]$. We have been given an initial value problem of the type

$$y'(t) = f(y(t)), \quad t \in \mathbb{R}$$

$$y(0) = y_0, \quad y_0 \in \mathbb{R}$$

and now we desire an approximation of $y(t)$ for all $t \in \Lambda$. In general, an approximation is almost useless on its own, and we must always find a reliable error estimate!

To this end we consider pick an integer $m > 0$ and define

$$h = \frac{\bar{h}}{4m}$$

Then we consider the grids

$$\Sigma_{4h} \subset \Sigma_{2h} \subset \Sigma_h$$

for which the choice of $h$ ensures that

$$\Lambda \subset \Sigma_{4h}.$$ 

Finally, we compute three grid functions $v_{4h}, v_{2h}, v_h$ using, say, our favorite Runge-Kutta method and carefully store the values

$$\{ (v_{h}(t), v_{2h}(t), v_{4h}(t)) \mid t \in \Lambda \}$$

Now, it turns out there are theorems of the following type for essentially any solver that we would like to use. The theorems are somewhat difficult to derive, but they are easy to verify numerically and we can exploit them to compute good error estimates.

**Theorem 1.** If $f$ is sufficiently smooth, then there exists a function $\alpha : \mathbb{R} \to \mathbb{R}$ and numbers $p < q$ such that

$$\forall t \in \Sigma_h : y(t) - v_h(t) = \alpha(t)h^p + O(h^q).$$

The numbers $p$ and $q$ depend on the numerical method and there are also complicated formulas for the function $\alpha$. However, we can still proceed exactly as in the case of numerical integration. We find

$$\forall t \in \Sigma_{2h} : \alpha(t)h^p = \frac{v_h(t) - v_{2h}(t)}{2^p - 1} + O(h^q)$$

which allows us estimate

$$\forall t \in \Lambda : y(t) - v_h(t) = \frac{v_h(t) - v_{2h}(t)}{2^p - 1} + O(h^q)$$

except for the small problem, that we do not necessarily know the appropriate value of $p$. However, it is not hard to see that if $\alpha(t) \neq 0$, then

$$\forall t \in \Lambda : \frac{v_{2h}(t) - v_{4h}(t)}{v_h(t) - v_{2h}(t)} \approx 2^p$$

provided that $h$ is suitably small. In short, we can identify the appropriate value of $p$ simply by monitoring a suitable fraction.