Image Analysis

Mathematical Morphology

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Repetition

But First...

Pixel relations

Given that the sampling has been carried ut with a square lattice, the following holds:

- A pixel p at coordinate (x, y) has four **horizontal and vertical neighbors**:
  $(x+1, y)$, $(x, y+1)$, $(x-1, y)$ and $(x, y-1)$
- This set of pixels is called $N_4(p)$
- The set of **diagonal neighbors** of p:
  $(x+1, y+1)$, $(x+1, y-1)$, $(x-1, y+1)$ and $(x-1, y-1)$ is called $N_D(p)$
- The set $N_4(p) \cup N_D(p)$ is called $N_8(p)$ or the **8-neighbors** of p.

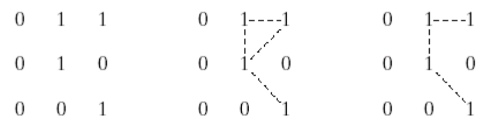
If two pixels p and q are neighbors in some perspective **and** $f(p) \in V$, $f(q) \in V$ for some $V \subset L$, you say that p and q are **adjacent**.

4-adjacency $p \in N_4(q)$ and $f(p) \in V$, $f(q) \in V$

8-adjacency $p \in N_8(q)$ and $f(p) \in V$, $f(q) \in V$

m-adjacency $f(p) \in V$, $f(q) \in V$ and

- $q \in N_4(p)$ or
- $q \in N_D(p)$ and $N_4(p) \cap N_4(q) = \emptyset$



- Two sets S_1 and S_2 are adjacent if $p \in S_1$ and $q \in S_2$ and p and q are adjacent.
- A **path** from p to q is a finite sequence of points $p_0, p_1, \ldots, p_n$ where p_i and p_{i+1} are adjacent and $p_0 = p$ and $p_n = q$. The path will be unique only if m-adjacency has been used.

- $p, q \in S$ are **connected** if there is a path from p to q where **all** points are in S.
- All pixels q that are connected to a pixel p is said to constitute a **connected component** (or an **object**)

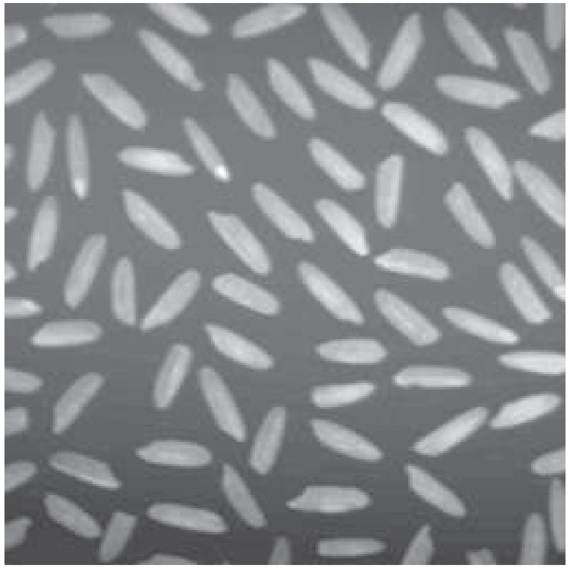
Mathematical Morphology

- Within biology, the term morphology is used for the study of the shape and structure of animals and plants
- In image processing mathematical morphology is theoretical framework for **representation**, **description** and **pre-processing**.
- The mathematical morphology is based on **set-theory**
- The mathematical morphology is an example of **non-linear operations**

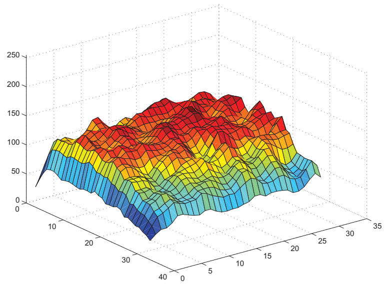
Linear filtering

Morphological filtering

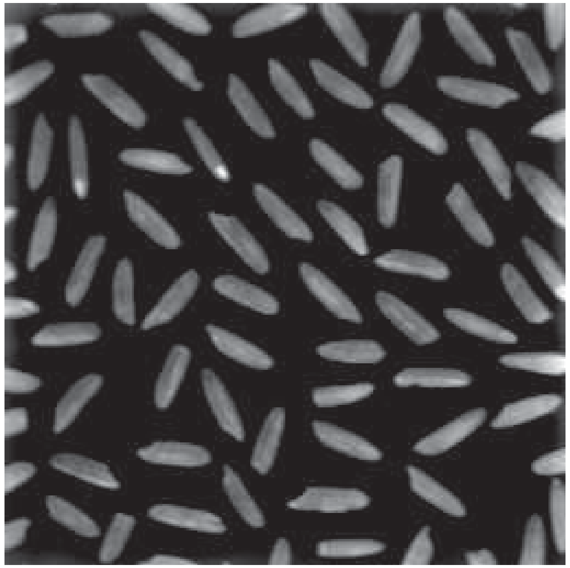
Median filtering



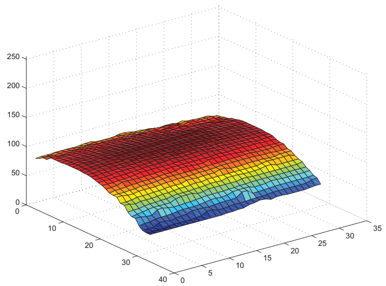
Low pass filtered image



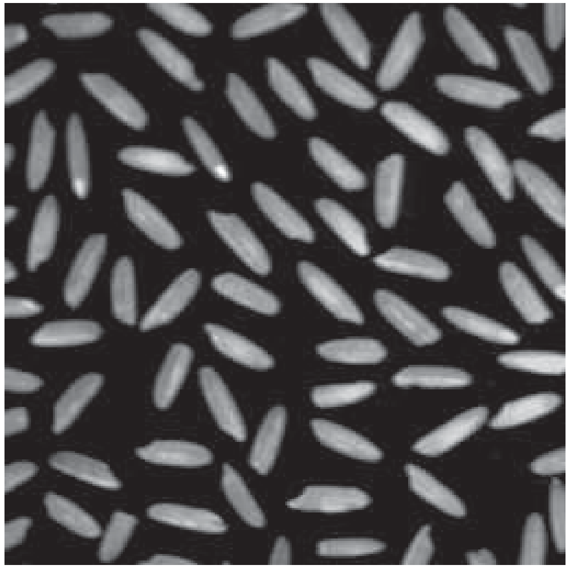
Background subtracted



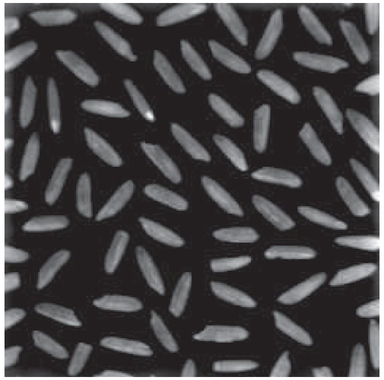
Morphological filtered background

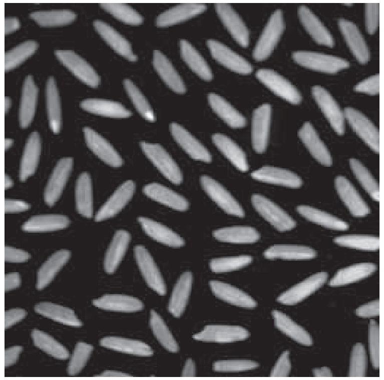


Background subtracted



Subtracted backgrounds - Linear and Morphological





Let A and B be sets of points from $\mathbb{Z}^2$ with objects $a = (a_1, a_2)$ and $b = (b_1, b_2)$.

The Translation of A with $x = (x_1, x_2)$ is defined as:

$$(A)_x = \{c \mid c = a + x, \forall a \in A\}$$

The Reflection of A is defined as:

$$\hat{A} = \{x \mid x = -a, a \in A\}$$

The Complement of A is defined as:

$$A^c = \{x \mid x \notin A\}$$

The Difference between two sets is defined as:

$$A - B = \{x \mid x \in A, x \notin B\} = A \cap B^c$$

The intersection and union of sets is supposed to be familiar

The Basics

- A set represents a shape in mathematical morphology
- Binary images is a subset of $\mathbb{Z}^2$ where each pixel is a tuple of coordinates (x, y) for either the black or white pixels.
- Intensity images is a subset of $\mathbb{Z}^3$ where each element is a three-dimensional points who's first two elements describe the spatial locality and the last element describes the intensity at that point.

Dilation

The Dilation of $A \subset \mathbb{Z}^2$ and $B \subset \mathbb{Z}^2$ is defined as

$$A \oplus B = \{x \mid (\hat{B})_x \cap A \neq \emptyset\}$$

The dilation is all translations x that yields an intersection between A and the reflected B that is not the empty set.

$\therefore$ dilation ensures that the intersection between A and $\hat{B}$ is a subset of A or

$$A \oplus B = \{x \mid [(\hat{B})_x \cap A] \subseteq A\}$$

B is called the **structuring element** of the dilation (as in all other morphological operations).

1	1	1
1	1	1
1	1	1

Original image

Dilated image

1	0	0
0	1	0
0	0	1

Original image

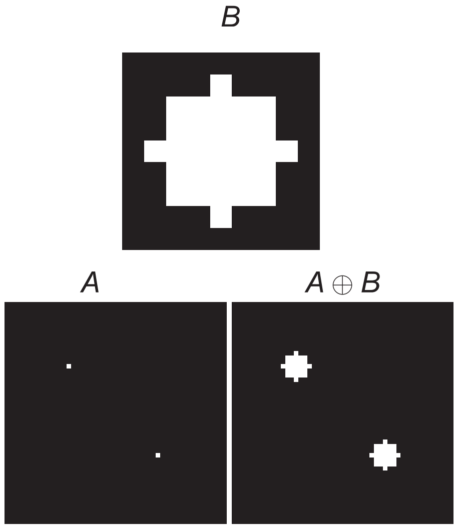
Dilated image

Another definition of the dilation is (*Minkowsky addition*)

$$A \oplus B = \bigcup_{b \in B} (A)_b$$

This is, of course, equivalent to the previous definition

The Minkowsky addition allows us to visualize the dilation as follows:
A copy of the structure element B is placed at each point $x \in A$.



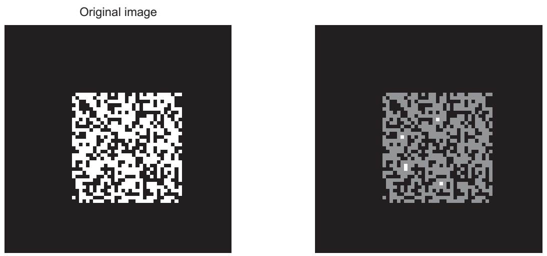
Erosion

The Erosion of $A \subset \mathbb{Z}^2$ and $B \subset \mathbb{Z}^2$ is defined as:

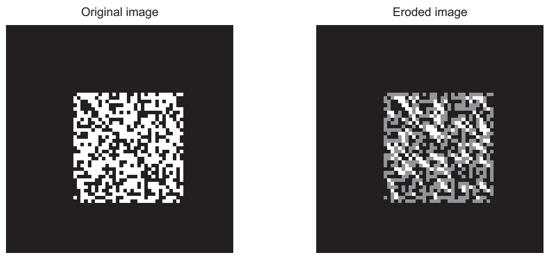
$$A \ominus B = \{x \mid (B)_x \subseteq A\}$$

In other words: The Erosion of A with B are the translation x in which B is completely covered by A.

1	1	1
1	1	1
1	1	1



1	0	0
0	1	0
0	0	1



Duality

Erosion and dilation are dual with respect to complement and reflection:

$$(A \ominus B)^c = A^c \oplus \hat{B}$$

Since (starting from definitions)

$$\begin{aligned} (A \ominus B)^c &= \{x \mid (B)_x \subseteq A\}^c && (1)\\ &= \{x \mid (B)_x \cap A^c = \emptyset\}^c && (2)\\ &= \{x \mid (B)_x \cap A^c \neq \emptyset\} && (3)\\ &= A^c \oplus \hat{B} && (4) \end{aligned}$$

Opening

- By combining erosion with dilation according to

$$A \circ B = (A \ominus B) \oplus B$$

  we have an operation that is called **opening**.
- The Opening smooths objects by **removing pixels**
- An other definition is

$$A \circ B = \bigcup\{(B)_x \mid (B)_x \subseteq A\}$$

Closing

- By reversing the erosion and dilation in the opening, a closing operation is defined:

$$A \bullet B = (A \oplus B) \ominus B$$

- Closing smooths objects by **adding pixels**.
- All smoothing (both for closing and opening) is in relation to the size of the structuring element B.
- Opening and closing are dual with respect to complement and reflection

For opening the following holds

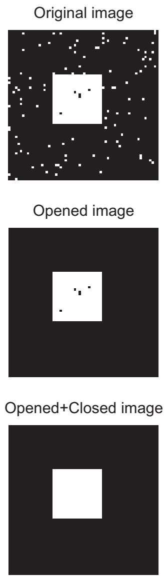
1. $A \circ B \subset A$
2. If $C \subset D$ then $C \circ B \subset D \circ B$
3. $(A \circ B) \circ B = A \circ B$

and for closing

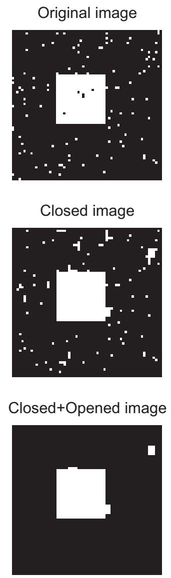
1. $A \subset A \bullet B$
2. If $C \subset D$ then $C \bullet B \subset D \bullet B$
3. $(A \bullet B) \bullet B = A \bullet B$

Operators for which property (3) holds are said to be **idempotent**. That is, it does not matter how many times the operation is applied, the result is the same as if it only had been applied once.

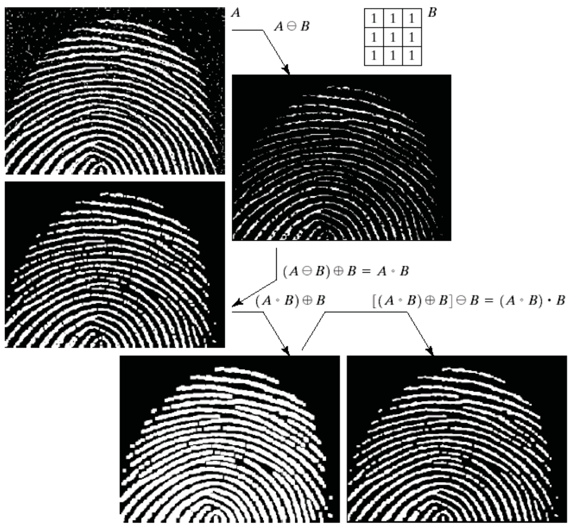
$(A \circ B) \bullet B$



$(A \bullet B) \circ B$



$(A \circ B) \bullet B$



Hit & Miss

If the aim is to locate objects D in an image we can do the following:

1. By calculating $A \ominus D$ all objects smaller than D will vanish. D is reduced to a point.
2. By calculating $A^c \ominus (W - D)$, where $W - D$ is a frame to the object, all objects larger than D will vanish, D is reduced to a point.

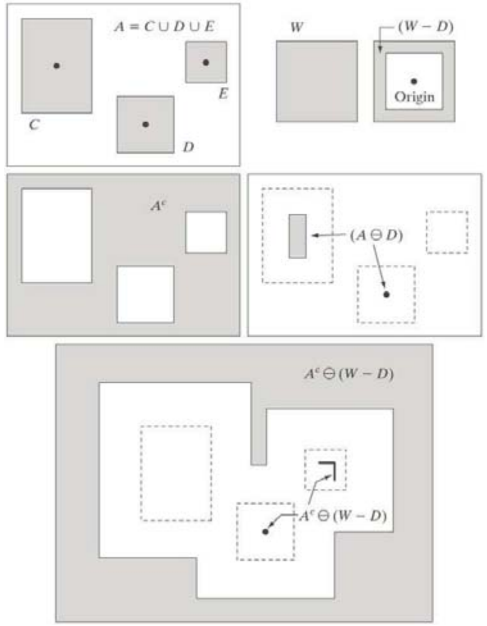
The intersection between (1) and (2) will be the mid-point for each D in the image.

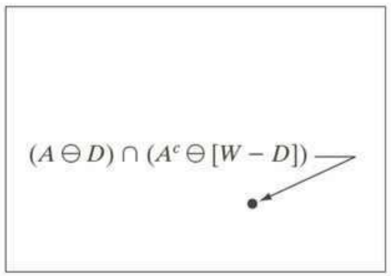
Or slightly more formal:

$$(A \ominus D) \cap [A^c \ominus (W - D)]$$

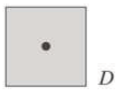
If we let $B = (B_1, B_2)$ and $B_1 = D$, $B_2 = W - D$ we have

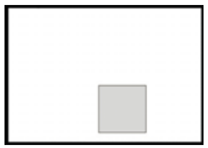
$$A \circledast B = (A \ominus B_1) \cap (A^c \ominus B_2)$$





$$((A \ominus D) \cap [A^c \ominus (W - D)]) \oplus D$$





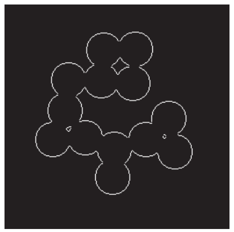
Edge extraction

The pixels that constitutes the edge of an object A is defined as the points that belong to the object and has at least one neighbor that belongs to the background.
These pixels are denoted $\beta(A)$, and are precisely the pixels that are removed by performing an erosion on A with a suitable structure element B,

$$\beta(A) = A - (A \ominus B)$$

Original image, A

A−(A eroded with B)



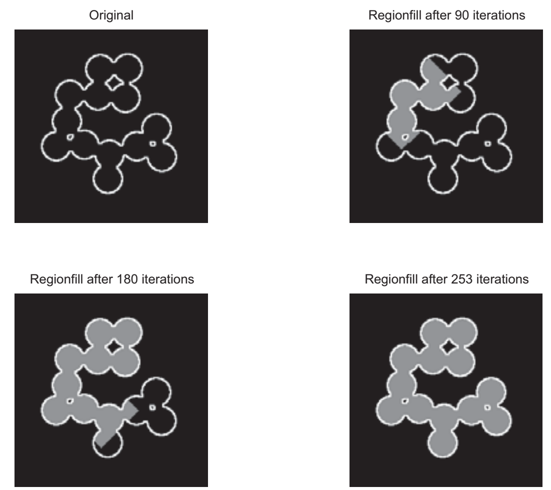
Region growing

Given that we know a point p in side a region defining with a contour A we can let the point grow until it fills the entire region by iterating with

$$X_k = (X_{k-1} \oplus B) \cap A^c \quad k = 1, 2, 3, \ldots$$

until $X_k = X_{k+1}$. X_0 is sat to be p.
If the region is 8-connected the complement will be 4-connected and B must have the following appearance

0	1	0
1	1	1
0	1	0



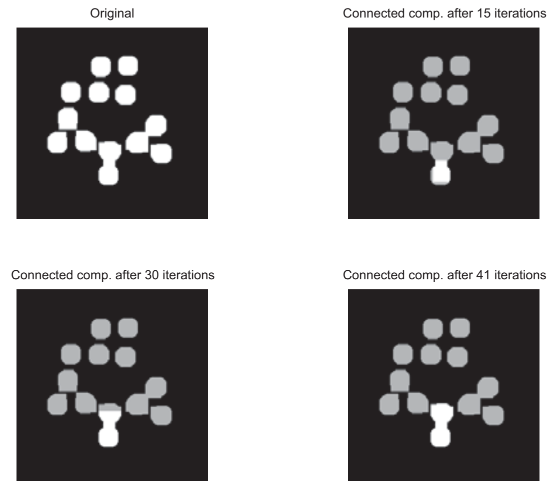
Connected components

Given a point p inside an object we can use the same principle as in region-growing with a slight modification. The difference is that since the object now is defined by ones (and not zeros surrounded by a contour) we do not have to invert A.

$$X_k = (X_{k-1} \oplus B) \cap A \quad k = 1, 2, 3, \ldots$$

where $X_0 = p$.
If A is 8-connected, the object will be 8-connected and we have to use the following B

1	1	1
1	1	1
1	1	1



Convex hull

- Given a set A we can find an **approximation** to the convex hull $C(A)$ by repeated application of

$$X_k^i = (X \circledast B^i) \cup A$$

where $i = 1, 2, 3, 4$ and $k = 1, 2, 3, \ldots$.
B^i are different structuring elements.

- For each i, $\{X_k^i\}$ will converge to a set D^i.
- The convex hull is the union of all D^i

$$C(A) = \bigcup_{i=1}^{4} D^i$$

- The following B^i are used in the algorithm

B^1:

1	×	×
1	0	×
1	×	×

B^2:

1	1	1
×	0	×
×	×	×

B^3:

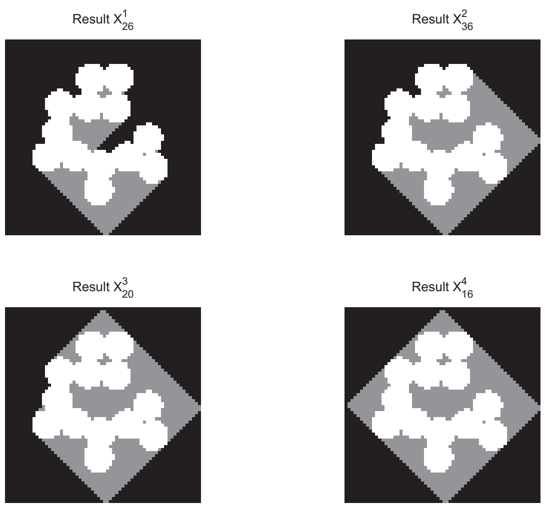
×	×	1
×	0	1
×	×	1

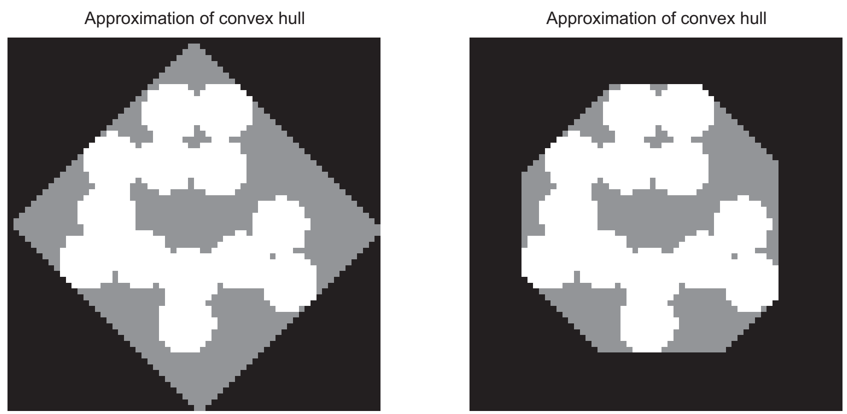
B^4:

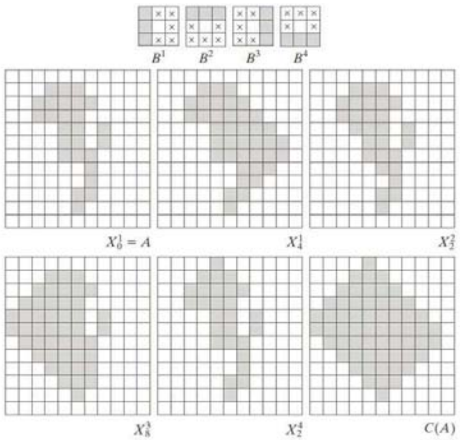
×	×	×
×	0	×
1	1	1

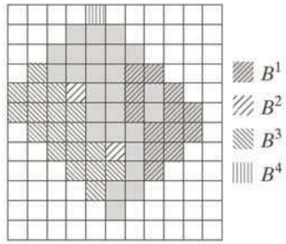
× indicates don't care.

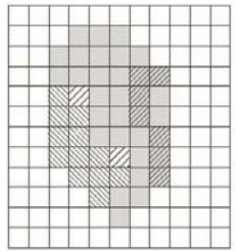
- This definition (with don't care) makes it impossible to use MatLabs implementation of dilation and erosion since they only use binary structuring elements.
- The convex hull is implemented in MatLab by using **look-up tables** (**LUT**), see `help makelut` and `help applylut` for more information.











The Skeleton

The skeleton $S(A)$ of an object A can be expressed as

$$S(A) = \bigcup_{k=0}^{K} S_k(A)$$

where

$$S_k(A) = (A \ominus kB) - ((A \ominus kB) \circ B)$$

and

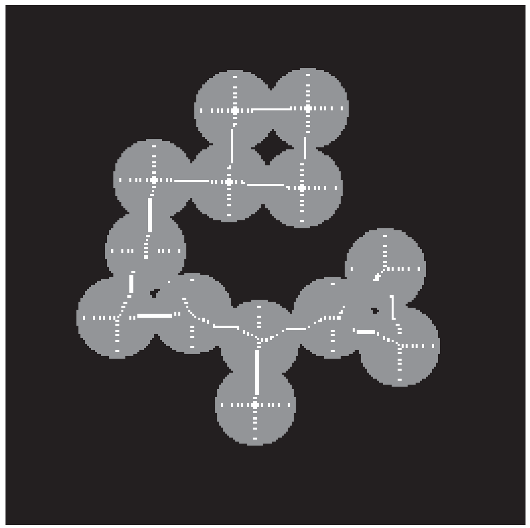
$$K = \max\{k \mid (A \ominus kB) \neq \emptyset\}$$

and finally

$$(A \ominus kB) = \underbrace{((\ldots((A \ominus B) \ominus B) \ominus \ldots) \ominus B}_{k \text{ times}}$$

Observe: There is nothing in the algorithm that guarantees that $S(A)$ will be a connected component.

An object and its skeleton



$K = 15$

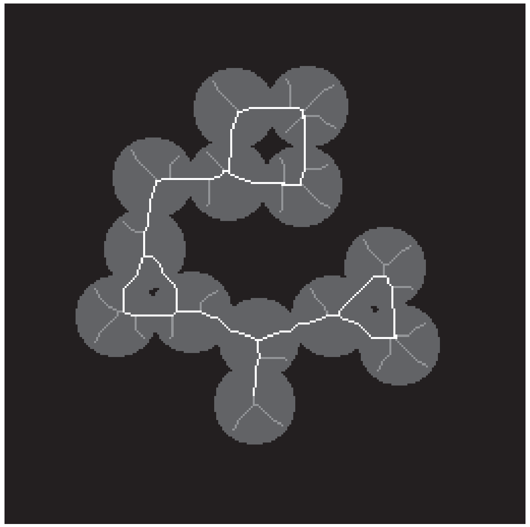
- A nice property of $S(A)$ is the A can be reconstructed if we know K.
- The reconstruction is made according to

$$A = \bigcup_{k=0}^{K} (S_k(A) \oplus kB)$$

Pruning

- After finding the skeleton, it is often desirable to remove short branches from the graph that constitutes the skeleton. This is done by removing end-points from the graph without breaking connectivity.
- Given knowledge of the object (that is how long branches that can be expected) we can remove just enough of the graph.

Image, skeleton and graph



Gray level

- To use mathematical morphology on gray-scale images the crisp set theory must be replaced by fuzzy set theory.
- This means that each point (x, y) has a **membership function** $f(x, y)$ describing the degree of membership that the point has to the foreground and background.
- $f(x, y)$ is the image, $b(x, y)$ is the structuring element. $f^c(x, y)$ is defined as $-f(x, y)$ and $\hat{b}(x, y)$ as $b(-x, -y)$.

Dilation A dilation of f with b is defined as

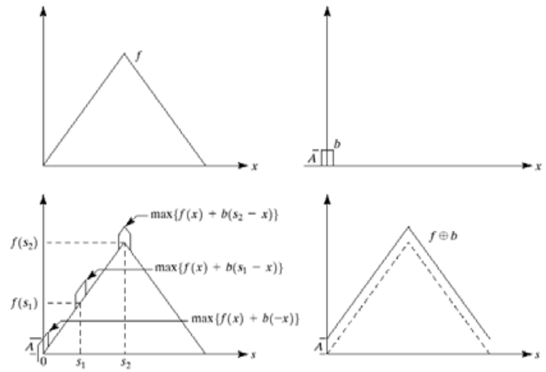
$$(f \oplus b)(s,t) = \max\{f(s-x,t-y) + b(x,y) \mid (s-x),(t-y) \in D_f, (x,y) \in D_b\}$$

Erosion The erosion of f with b is defined as

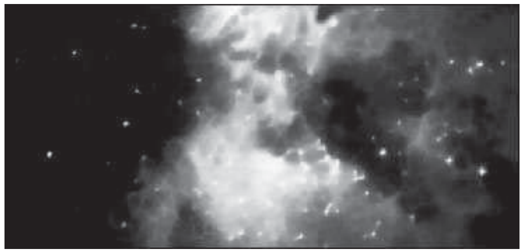
$$(f \oplus b)(s,t) = \min\{f(s-x,t-y) - b(x,y) \mid (s-x),(t-y) \in D_f, (x,y) \in D_b\}$$

D_f and D_b are the domain for f and b respectively. Since $(x,y) \in D_b$ only the points in the support of f and b can be members of the final set. Dilation and erosion are still dual :

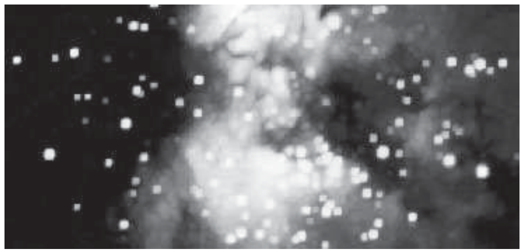
$$(f \ominus b)^c(x,y) = (f^c \oplus \hat{b})(x,y)$$



Eroded image



Dilated image



- **The Opening** of f with b is defined as

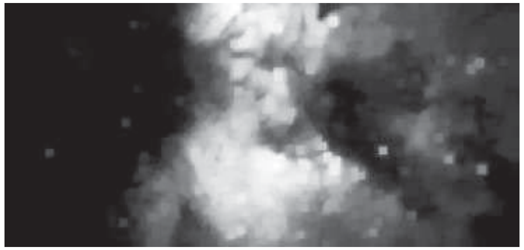
$$(f \circ b)(s,t) = ((f \oplus b) \ominus b)(s,t)$$

- The opening suppress bright objects smaller than the structuring element b
- **The Closing** of f with b is defined as

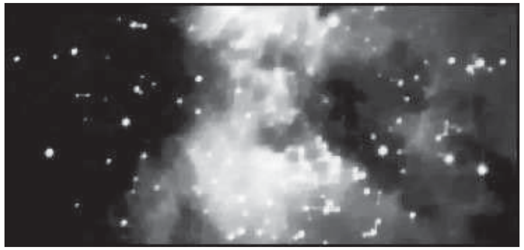
$$(f \bullet b)(s,t) = ((f \ominus b) \oplus b)(s,t)$$

- The closing suppress dark objects smaller than the structuring element b.

Opened image



Closed image



- A morphological **low-pass filtering** can be calculated as

$$l = (f \circ b) \bullet b$$

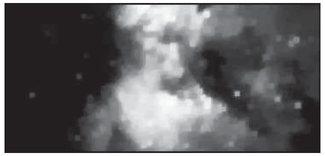
- A morphological **gradient filtering** can be calculated as

$$g = (f \oplus b) - (f \ominus b)$$

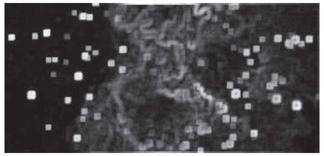
- A **'top-hat' transform** is defined as

$$h = f - (f \circ b)$$

Opened and closed image



Morphological gradient



Top−Hat