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Today’s Menu

1. Different Types of Reductions
2. Reduction by Restriction
3. Reduction by Local Replacement
4. Reduction by Composition of Gadgets
Different Types of Reductions

Types of Reductions

Notation: NPC denotes the class of all NP-complete problems.

Suppose we want to show that \( A \in \text{NPC} \). If we already know that \( A \in \text{NP} \), we have to find a problem \( B \in \text{NPC} \) and a polynomial-time reduction from \( B \) to \( A \).

The three most common types of reductions:

1. reduction by restriction (simple)
2. reduction by local replacement (still usually rather simple)
3. reduction by composition of “gadgets” (can be quite tricky)
Idea: If $A$ is a more general variant of a problem $B \in \text{NPC}$, then the reduction only has to turn an instance of $B$ into an instance of $A$.

Intuition: If $A$ is more general than $B$, then $A$ cannot be easier than $B$. 

Reduction by Restriction
Recall Hamiltonian Cycle (HAM)

**Input:** An undirected graph $G = (V, E)$.

**Question:** Does $G$ contain a simple cycle of length $|V|$?

Directed Hamiltonian Cycle (dHAM) is defined in the same way, except that $G$ is directed (and a directed simple cycle of length $|V|$ is sought).

Assume that we already know that HAM $\in$ NPC.

**In which sense is dHAM a more general variant of HAM?**
Example: Reducing HAM to dHAM

Reduction $f$ from HAM to dHAM: In the undirected input graph $G$, turn every edge $\bullet \longrightarrow \bullet$ into two antiparallel edges: $\bullet \overset{\rightarrow}{\longrightarrow} \bullet$.

Correctness:

- Computability of $f$ in polynomial time is obvious.
- If $G \in \text{HAM}$, then it has a Hamiltonian cycle $v_0 \cdots v_n$. Thus, $(v_{i-1}, v_i)$ is a directed edge in $f(G)$ for all $i \in \{1, \ldots, n\}$, which means that $v_0 \cdots v_n$ is a directed cycle in $f(G)$, i.e., $f(G) \in \text{dHAM}$.
- If $f(G) \in \text{dHAM}$, then it has a directed Hamiltonian cycle $v_0 \cdots v_n$. Since $f(G)$ contains an directed edge $(v_{i-1}, v_i)$ only if $G$ contains the corresponding undirected edge, this means that $v_0 \cdots v_n$ is a Hamiltonian cycle in $G$. In other words, $G \in \text{HAM}$.
Example: Reducing HAM to TSP

One version of the Travelling Salesman Decision Problem (TSP)

Input: An $n \times n$-matrix of distances $d_{i,j} \in \mathbb{N}$ and a number $k \in \mathbb{N}$.
Question: Is there a tour $v_0, \ldots, v_{n-1}$ such that 
$$\{v_0, \ldots, v_{n-1}\} = \{1, \ldots, n\} \text{ and } \sum_{j=1}^{n} d_{v_{j-1}, v_{j \mod n}} \leq k?$$

Assume again that we already know that HAM $\in$ NPC.

In which sense is TSP a more general variant of HAM?
Example: Reducing HAM to TSP

Reduction $f$ from HAM to TSP: Let $V = \{1, \ldots, n\}$ be the set of nodes of the input graph $G$. Let $k = n$ and, for $i, j \in \{1, \ldots, n\}$,

$$d_{i,j} = \begin{cases} 1 & \text{if } G \text{ contains the edge } (i, j) \\ 2 & \text{otherwise.} \end{cases}$$

Correctness:

- Computability of $f$ in polynomial time is again obvious.
- If $G \in \text{HAM}$, then it has a Hamiltonian cycle $v_0 \cdots v_n$. Thus, $d_{v_{j-1}, v_{j \mod n}} = 1$ for all $i, j \in \{1, \ldots, n\}$, which means that the tour $v_0 \cdots v_{n-1}$ has length $n = k$.
- If $f(G) \in \text{TSP}$, then there is a tour $v_0 \cdots v_{n-1}$ of length $k \leq n$. Since the only distances are 1 and 2, this means that all the distances on this tour are 1. Thus, $v_0 \cdots v_{n-1}v_0$ is a Hamiltonian cycle in $G$. 

Reduction by Local Replacement

**Idea:** To turn an instance of $B \in \text{NPC}$ into a corresponding instance of $A$, we locally replace substructures of an instance of $B$ by other substructures.

This is often used in order to **turn a more general problem into a special form**, showing that even this special form is NP-complete.
Recall (?) 3-Satisfiability (3SAT)

**Input:** An propositional formula \( \varphi \) in CNF in which each clause has exactly 3 literals.

**Question:** Is \( \varphi \) satisfiable?

We already know that \( \text{SAT} \in \text{NPC} \).

How can we turn an instance of \( \text{SAT} \) into an equivalent instance of \( 3\text{SAT} \)?
Example: Reducing SAT to 3SAT

Reduction \( f \) from SAT to 3SAT: replace every individual clause \((l_1 \lor \cdots \lor l_k)\) by several clauses consisting of 3 literals each.

\[
\begin{align*}
(l) & \implies (l \lor y_1 \lor y_2) \land \\
 & \land (l \lor y_1 \lor \neg y_2) \land \\
 & \land (l \lor \neg y_1 \lor y_2) \land \\
 & \land (l \lor \neg y_1 \lor \neg y_2) \\

(l_1 \lor l_2) & \implies (l_1 \lor l_2 \lor y_1) \land (l_1 \lor l_2 \lor \neg y_1) \\

(l_1 \lor \cdots \lor l_k) & \implies (l_1 \lor l_2 \lor y_1) \land \\
 & \land (\neg y_1 \lor l_3 \lor y_2) \land \\
 & \vdots \\
 & \land (\neg y_{k-4} \lor l_{k-2} \lor y_{k-3}) \land \\
 & \land (\neg y_{k-3} \lor l_{k-1} \lor l_k)
\end{align*}
\]
Example: Reducing SAT to 3SAT

Correctness of \( C = (l_1 \lor \cdots \lor l_k) \mapsto (l_1 \lor l_2 \lor y_1) \land \) 
\( (\neg y_1 \lor l_3 \lor y_2) \land \) 
\[ \vdots \]
\( (\neg y_{k-4} \lor l_{k-2} \lor y_{k-3}) \land \) 
\( (\neg y_{k-3} \lor l_{k-1} \lor l_k) = C' \)

- \( \alpha(C') = true \) for an assignment \( \alpha \)
  \( \Rightarrow \alpha(l_i) = true \) for some \( i \)
  \( \Rightarrow \) extending \( \alpha \) by \( \alpha(y_1) = \cdots = \alpha(y_{i-2}) = true \) and
  \( \alpha(y_{i-1}) = \cdots = \alpha(y_{k-3}) = false \) yields \( \alpha(C') = true \).

- \( \alpha(C'') = true \) for an assignment \( \alpha \)
  Consider the first clause that does not contain a \( y_i \) with \( \alpha(y_i) = true \)
  \( \Rightarrow \) this clause does not contain \( \neg y_{i-1} \) with \( \alpha(\neg y_{i-1}) = true \) either
  \( \Rightarrow \) the clause contains \( l_j \) with \( \alpha(l_j) = true \)
  \( \Rightarrow \alpha(C') = true. \)
Reduction by Composition of Gadgets

Gadgets are often used if the target of the reduction is a graph problem.

Idea: To turn an instance of \( B \in \text{NPC} \) into a corresponding instance of \( A \), we build an instance of \( A \) by composing copies of one or more “gadgets” that are constructed to fulfill a specific purpose.

Intuition: Think of composing, i.e., a binary adder using logical gates, or composing an 8-bit adder from 7 binary adders.
Example: Reducing SAT to dHAMPATH

Recall (?) directed Hamiltonian Cycle (dHAMPATH)

Input: A directed graph \( G = (V, E) \).

Question: Does \( G \) contain a simple path of length \(|V|\)?

How can we turn an instance of SAT into an equivalent instance of dHAMPATH?

In the following, consider a formula \( \varphi \) in CNF with \( n \) variables \( x_1, \ldots, x_n \). Let \( l_2, l_4, \ldots, l_m \) be the sequence of literals in (the clauses of) \( \varphi \).
A gadget for choosing the truth value of $x_i$:

A path will enter the gadget at $v_1^i$ and leave it at $v_m^i$ if $\alpha(x_i) = true$ and conversely if $\alpha(x_i) = false$.

The gadget contains twice as many nodes as $\varphi$ contains literals.

The gadget for the clause containing $l_j$ will be attached to $v_{j-1}^i$ and $v_j^i$ if $l_j \in \{x_i, \neg x_i\}$.

Notes:
The gadget for the clauses:

\[
\ldots x_i \ldots \quad \text{or} \quad \ldots \neg x_i \ldots 
\]  
(clause with literal \( l_j \))

Notes:

- One extra node per clause \( C \).
- If \( l_j = x_i \) or \( l_j = \neg x_i \) is in \( C \), it is connected to \( v_{j-1}^i, v_j^i \) as shown.
- Passing the \( x_i \)-gadget left to right, we can make a “detour” to pass \( C \) if the literal \( l_j = x_i \) occurs in \( C \).
- Similarly when passing from right to left and \( l_j = \neg x_i \) occurs in \( C \).
Putting it all together:

\[
x_1 \text{-gadget} \\
\vdots \\
x_n \text{-gadget}
\]

\[
C_1 \quad \cdots \quad C_k
\]
Example: Reducing SAT to dHAMPATH

Correctness, direction 1: Suppose $\alpha$ makes $\varphi$ true.

- Start at $\text{start}$.
- Continue to $v^1_1$ if $\alpha(x_1) = \text{true}$; otherwise, go to $v^1_m$.
- Pass through the $x_1$-gadget with detours via clause gadgets (see below).
- Continue similarly with the $x_2$-gadget, and so on.
- Detours when passing the $x_i$-gadget from left to right ($\alpha(x_i) = \text{true}$): If
  - a clause $C$ contains the literal $l_j = x_i$ and
  - the node corresponding to $C$ has not yet been passed
  go from $v^i_{j-1}$ to $v^i_j$ via $C$ (rather than directly from $v^i_{j-1}$ to $v^i_j$).
- Similarly if $\alpha(x_i) = \text{false}$ and $C$ contains the literal $l_j = \neg x_i$.

As $\alpha(C) = \text{true}$ for every clause $C$, this visits each node once.
Example: Reducing SAT to dHAMPATH

Correctness, direction 2: Suppose \( f(\varphi) \in \text{dHAMPATH} \).

- The path must start at \text{start}, pass the \( x_i \)-gadgets one after another, and end at \text{end}.

- In particular, clause nodes can only be entered and left via “companion nodes” \( v_{j-1}^{i}, v_{j}^{i} \) (otherwise, there is no way to return).

- Depending on the direction in which the \( x_i \)-gadgets are passed, this yields a truth assignment \( \alpha \).

- Since all the clause nodes are on the paths, they must have been included via detours.

- A detour is only possible if the corresponding literal is made \text{true} by \( \alpha \).

Hence, \( \alpha(C) = \text{true} \) for all clauses \( C \), meaning that \( \alpha(\varphi) = \text{true} \).
Please read the remainder of Chapter 10 in the textbook.