Efficient Algorithms and Problem Complexity
– Techniques for Constructing Reductions –

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Today’s Menu

1. Different Types of Reductions
2. Reduction by Restriction
3. Reduction by Local Replacement
4. Reduction by Composition of Gadgets
Types of Reductions

Notation: NPC denotes the class of all NP-complete problems.

Suppose we want to show that $A \in NPC$. If we already know that $A \in NP$, we have to find a problem $B \in NPC$ and a polynomial-time reduction from $B$ to $A$.

The three most common types of reductions:

1. reduction by restriction (simple)
2. reduction by local replacement (still usually rather simple)
3. reduction by composition of “gadgets” (can be quite tricky)
Reduction by Restriction

Idea: If $A$ is a more general variant of a problem $B \in \text{NPC}$, then the reduction only has to turn an instance of $B$ into an instance of $A$.

Intuition: If $A$ is more general than $B$, then $A$ cannot be easier than $B$. 
Example: Reducing HAM to dHAM

Recall Hamiltonian Cycle (HAM)

Input: An undirected graph $G = (V, E)$.

Question: Does $G$ contain a simple cycle of length $|V|$?

Directed Hamiltonian Cycle (dHAM) is defined in the same way, except that $G$ is directed (and a directed simple cycle of length $|V|$ is sought).

Assume that we already know that HAM $\in$ NPC.

In which sense is dHAM a more general variant of HAM?
Example: Reducing HAM to dHAM

Reduction $f$ from HAM to dHAM: In the undirected input graph $G$, turn every edge into two antiparallel edges.

Correctness:

- Computability of $f$ in polynomial time is obvious.
- If $G \in \text{HAM}$, then it has a Hamiltonian cycle $v_0 \cdots v_n$. Thus, $(v_{i-1}, v_i)$ is a directed edge in $f(G)$ for all $i \in \{1, \ldots, n\}$, which means that $v_0 \cdots v_n$ is a directed cycle in $f(G)$, i.e., $f(G) \in \text{dHAM}$.
- If $f(G) \in \text{dHAM}$, then it has a directed Hamiltonian cycle $v_0 \cdots v_n$. Since $f(G)$ contains an directed edge $(v_{i-1}, v_i)$ only if $G$ contains the corresponding undirected edge, this means that $v_0 \cdots v_n$ is a Hamiltonian cycle in $G$. In other words, $G \in \text{HAM}$. 
Example: Reducing HAM to $\text{TSP}_D$

One version of the Travelling Salesman Decision Problem ($\text{TSP}_D$)

**Input:** An $n \times n$-matrix of distances $d_{i,j} \in \mathbb{N}$ and a number $k \in \mathbb{N}$.

**Question:** Is there a tour $v_0, \ldots, v_{n-1}$ such that \[ \{v_0, \ldots, v_{n-1}\} = \{1, \ldots, n\} \] and \[ \sum_{j=1}^{n} d_{v_{j-1}, v_j \mod n} \leq k? \]

Assume again that we already know that $\text{HAM} \in \text{NPC}$.

In which sense is $\text{TSP}_D$ a more general variant of $\text{HAM}$?
Example: Reducing HAM to TSP<sub>D</sub>

Reduction \( f \) from HAM to TSP<sub>D</sub>: Let \( V = \{1, \ldots, n\} \) be the set of nodes of the input graph \( G \). Let \( k = n \) and, for \( i, j \in \{1, \ldots, n\} \),

\[
d_{i,j} = \begin{cases} 
1 & \text{if } G \text{ contains the edge } (i, j) \\
2 & \text{otherwise}.
\end{cases}
\]

Correctness:

- Computability of \( f \) in polynomial time is again obvious.
- If \( G \in \text{HAM} \), then it has a Hamiltonian cycle \( v_0 \cdots v_n \). Thus, \( d_{v_{j-1}, v_{j \mod n}} = 1 \) for all \( i, j \in \{1, \ldots, n\} \), which means that the tour \( v_0 \cdots v_{n-1} \) has length \( n = k \).
- If \( f(G) \in \text{TSP}_D \), then there is a tour \( v_0 \cdots v_{n-1} \) of length \( \leq n \). Since the only distances are 1 and 2, this means that all the distances on this tour are 1. Thus, \( v_0 \cdots v_{n-1}v_0 \) is a Hamiltonian cycle in \( G \).
Reduction by Local Replacement

Idea: To turn an instance of $B \in \text{NPC}$ into a corresponding instance of $A$, we locally replace substructures of an instance of $B$ by other substructures.

This is often used in order to turn a more general problem into a special form, showing that even this special form is NP-complete.
Example: Reducing SAT to 3SAT

Recall (?) 3-Satisfiability (3SAT)

Input: A propositional formula $\varphi$ in CNF in which each clause has exactly 3 distinct literals.

Question: Is $\varphi$ satisfiable?

We already know that $\text{SAT} \in \text{NPC}$.

How can we turn an instance of $\text{SAT}$ into an equivalent instance of $\text{3SAT}$?

We replace every clause by a bunch of clauses consisting of 3 literals each.
Example: Reducing SAT to 3SAT

Replacing a clause \((l_1 \lor \cdots \lor l_k)\) by clauses consisting of 3 literals each, using new variables \(y_i\):

\[
\begin{align*}
(l) & \mapsto (l \lor y_1 \lor y_2) \land \\
&(l \lor y_1 \lor \neg y_2) \land \\
&(l \lor \neg y_1 \lor y_2) \land \\
&(l \lor \neg y_1 \lor \neg y_2)
\end{align*}
\]

\[
\begin{align*}
(l_1 \lor l_2) & \mapsto (l_1 \lor l_2 \lor y_1) \land (l_1 \lor l_2 \lor \neg y_1)
\end{align*}
\]

\[
\begin{align*}
(l_1 \lor \cdots \lor l_k) & \mapsto (l_1 \lor l_2 \lor y_1) \land \\
&(\neg y_1 \lor l_3 \lor y_2) \land \\
&(\neg y_2 \lor l_4 \lor y_3) \land \\
&\quad \vdots \\
&(\neg y_{k-4} \lor l_{k-2} \lor y_{k-3}) \land \\
&(\neg y_{k-3} \lor l_{k-1} \lor l_k)
\end{align*}
\]
Example: Reducing SAT to 3SAT

Correctness of $C = (l_1 \lor \cdots \lor l_k) \implies (l_1 \lor l_2 \lor y_1) \land (y_1 \lor l_3 \lor y_2) \land \cdots \land (y_{k-4} \lor l_{k-2} \lor y_{k-3}) \land (y_{k-3} \lor l_{k-1} \lor l_k) = C'$

- $\alpha(C') = true$ for an assignment $\alpha$
  - $\implies \alpha(l_i) = true$ for some $i$
  - $\implies$ extending $\alpha$ by $\alpha(y_1) = \cdots = \alpha(y_{i-2}) = true$ and $\alpha(y_{i-1}) = \cdots = \alpha(y_{k-3}) = false$ yields $\alpha(C') = true$.

- $\alpha(C'') = true$ for an assignment $\alpha$
  - Consider the first clause that does not contain a $y_i$ with $\alpha(y_i) = true$
  - $\implies$ this clause does not contain $\neg y_{i-1}$ with $\alpha(\neg y_{i-1}) = true$ either
  - $\implies$ the clause contains $l_j$ with $\alpha(l_j) = true$
  - $\implies \alpha(C') = true.$
Reduction by Composition of Gadgets

Gadgets are often used if the target of the reduction is a graph problem.

Idea: Given an instance of $B \in \text{NPC}$ we build a corresponding instance of $A$ by composing copies of one or more “gadgets”.

The gadgets are building blocks constructed to fulfill a specific purpose.

Intuition: Think of composing, e.g., a binary adder using logical gates, or composing an 8-bit adder from 7 binary adders.
Example: Reducing SAT to dHAMPATH

Recall (?) directed Hamiltonian Path (dHAMPATH)

Input: A directed graph $G = (V, E)$.

Question: Does $G$ contain a simple path of length $|V| - 1$?

We want to reduce SAT to dHAMPATH.

How can we turn an instance of SAT into an equivalent instance of dHAMPATH?

Consider a formula $\varphi$ in CNF with $n$ variables $x_1, \ldots, x_n$. Let $l_1, \ldots, l_m$ be the sequence of literals in (the clauses of) $\varphi$. 
A gadget for **choosing the truth value of** $x_i$:

- Going left to right means $\alpha(x_i) = true$, right to left means $\alpha(x_i) = false$.
- Each literal is represented by four nodes.
- Another type of gadget will be used for the clauses.
The gadgets for the clauses attach to the variable clauses:

\[ C = \ldots x_i \ldots \] or \[ C = \ldots \neg x_i \ldots \]

- One extra node per clause \( C \).
- Passing the \( x_i \)-gadget left to right, we can make a “detour” to pass \( C \) if the literal \( l_j = x_i \) occurs in \( C \).
- Similarly when passing from right to left and \( l_j = \neg x_i \) occurs in \( C \).
Putting it all together:

\[
\begin{align*}
C_1 & \quad \ldots \quad C_k \\
\text{start} & \quad \ldots \\
\text{end} & \quad \ldots
\end{align*}
\]

\[x_1\text{-gadget} \quad \ldots \quad x_n\text{-gadget}\]
Example: Reducing SAT to dHAMPATH

Correctness, direction 1: Suppose $\alpha$ makes $\varphi$ true.

- Start at start.
- Go to left entry of $x_1$-gadget if $\alpha(x_1) = true$; otherwise, go to right entry.
- Pass through the $x_1$-gadget with detours via clause gadgets (see below).
- Continue similarly with the $x_2$-gadget, and so on.
- Detours when passing the $x_i$-gadget left to right ($\alpha(x_i) = true$): If
  - $C$ contains the literal $l_j = x_i$ and
  - $\star$ (the node corresponding to) $C$ had not been passed earlier
  then make a detour from $l_j$ (rather than passing straight through).
- Similarly if $\alpha(x_i) = false$ and $C$ contains the literal $l_j = \neg x_i$.

As $\alpha(C') = true$ for every clause $C'$, this visits each node once.
Example: Reducing SAT to dHAMPATH

Correctness, direction 2: Suppose $f(\varphi) \in dHAMPATH$.

- The path must start at start, pass the $x_i$-gadgets one after another, and end at end.
- In particular, if a clause node is entered from $l_j$, the path must immediately return to $l_j$ (otherwise, a "dead end" is created).
- Depending on the direction in which the $x_i$-gadgets are passed, this yields a truth assignment $\alpha$.
- Since all the clause nodes are on the paths, they must have been included via detours.
- A detour is only possible if the corresponding literal is made true by $\alpha$.

Hence, $\alpha(C) = true$ for all clauses $C$, meaning that $\alpha(\varphi) = true$. 
Please read and understand the NP-completeness proofs in any of the two textbooks (or in another book).