Review: the Finite Volume Method

Martin Berggren

September 10, 2015

Finite volume methods

- Originally developed for the solution of hyperbolic conservation laws, particularly gas dynamics
- Can robustly handle nonlinear phenomena like shocks and rarefaction waves
- Can handle complicated computational domain in 2D and 3D
- Dominating technique for high-speed flow (gas turbines, rockets, airplanes)
- Common technique for Computational Fluid Dynamics (CFD)
- Close “relative” to Finite Element Methods (FEM)

Finite volume methods for 1D conservation laws

Based on the **integral form** of the equation

Differential form (conservative):

\[ u_t + f(u)_x = 0 \]

Integrate in space \((a < b)\):

\[ \frac{d}{dt} \int_a^b u \, dx + f(u(b, t)) - f(u(a, t)) = 0 \]

Also integrating in time \((t_- < t_+)\):

\[
\int_a^b u(x, t_+) \, dx - \int_a^b u(x, t_-) \, dx
+ \int_{t_-}^{t_+} f(u(b, t)) \, dt - \int_{t_-}^{t_+} f(u(a, t)) \, dt = 0
\]

Subdivide the \(x\)-axis in **computational cells** of width \(\Delta x\)

and the consider discrete time levels \(t_0, t_1, \ldots\), at \(\Delta t\) distance.

Choose \(a = x_{i-1/2}, b = x_{i+1/2}, t_- = t_n, t_+ = t_{n+1}\).
Finite volume methods for 1D conservation laws

\[
\frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} u(x, t_{n+1}) \, dx = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} u(x, t_n) \, dx + \frac{\Delta t}{\Delta x} \left( \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} f(u(x_{i+1/2}, t)) \, dt - \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} f(u(x_{i-1/2}, t)) \, dt \right)
\]

Motivates introduction of a family of explicit, finite-volume schemes:

\[
u_{i+1}^{n+1} = u_i^n - \frac{\Delta t}{\Delta x} \left( F_{i+1/2}^n - F_{i-1/2}^n \right)
\]

where

\[
u_i^n \approx \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} u(x, t_n) \, dx
\]

\[
F_{i+1/2}^n \approx \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} f(u(x_{i+1/2}, t)) \, dt
\]

\(F_{i+1/2}^n\): numerical flux function. Defines the particular method.

The Lax–Friedrich scheme

Choosing the flux function

\[
F_{i+1/2}^n = \frac{1}{2} \left[ f(u_{i+1}^n) + f(u_i^n) \right] - \frac{\Delta x}{2\Delta t} [u_{i+1}^n - u_i^n]
\]

yields the scheme

\[
u_{i+1}^{n+1} = \frac{1}{2} (u_{i+1}^n + u_{i-1}^n) - \frac{\Delta t}{2\Delta x} \left[ f(u_{i+1}^n) - f(u_{i-1}^n) \right]
\]

The upwind and Lax–Friedrich schemes behave similarly:

- Very robust and stable
- Only first-order accurate: require very small \(\Delta t, \Delta x\) for accurate resolution of smooth solutions
- Can handle discontinuous solutions without oscillations, but tend to smear out the solution and reduce sharp spatial gradients. (The schemes are highly “dissipative”).

The upwind method

Recall: \(u_t + f(u)_x = u_t + f'(u)u_x = 0\). Thus:

\(f'(u) > 0\): transport to the right
\(f'(u) < 0\): transport to the left

Motivates the choice

\[
F_{i+1/2} = \begin{cases} f(u_{i+1}^n) & \text{if } f'(u) > 0, \\ f(u_i^n) & \text{if } f'(u) < 0 \end{cases}
\]

The flux function is evaluated in the "upwind direction."

Yields the scheme:

\[
u_{i+1}^{n+1} = \begin{cases} u_i^n - \frac{\Delta t}{\Delta x} (f(u_{i+1}^n) - f(u_{i-1}^n)) & \text{if } f' > 0, \\ u_i^n - \frac{\Delta t}{\Delta x} (f(u_{i+1}^n) - f(u_i^n)) & \text{if } f' < 0 \end{cases}
\]

Second-order-accurate methods

In lab 1 we tested a second-order method: the Richtmyer two-step Lax–Wendroff method

- Performs much better for smooth solutions
- However, tends to generate oscillations around discontinuities. (The scheme is “dispersive”).

More advanced methods: “high-resolution methods”

- Second-order accurate (or better) in smooth regions of the solution
- A limiter or artificial dissipation used in the cells around discontinuities to avoid oscillations. The scheme typically reduces to an upwind-like scheme around the shock
- The scheme becomes nonlinear (the scheme depends on the solution)! Needs “sensors” that detects regions of sharp gradients.
The CFL condition

A necessary condition for stability:
The characteristics through the “update” point must pass through the numerical domain of dependency (the gray region)

From picture, for schemes involving points $u^n_{i-1}$, $u^n_i$, and $u^n_{i+1}$,

$$\frac{\Delta t}{\Delta x} \leq \frac{1}{c}$$

Thus, it is necessary that $\Delta t \leq \Delta x/c$. (Note that the dimensions match!)