Kinematics 3: Kinematic Models of Sensors and Actuators
1 Vehicle Attitude in Euler Angle Form

1.1 Axis Conventions

- In aerospace vehicles, z points downward. Good for airplanes and satellites.
- Here z points up, y forward, and x out the right side. This has the advantage that the projection of 3D information onto the x-y plane is more natural.
- The convention used here corresponds to a z-x-y Euler angle sequence.
- It is not advisable to use the homogeneous transforms developed here until they are verified to be correct for any sensors and actuators that are used.

1.2 Frame Assignment

- Some common frames are indicated in the figure below:
1.2 Frame Assignment

### 1.2.1 The Navigation Frame

- Coordinate system in which the vehicle position and attitude is ultimately required.
- “Normally”, The z axis is aligned with the gravity vector; the y, or north axis is aligned with the geographic pole\(^1\); and the x axis points east to complete a right-handed system.

### 1.2.2 The Body Frame

- Positioned at the point on the vehicle body which is most convenient and is considered to be fixed in attitude with respect to the vehicle body.

### 1.2.3 The Positioner Frame

- Positioned at the point on or near any position estimation system which reports its own position.
- If the positioner system generates attitude and attitude rates only, this frame is not required because the attitude of the device will also be that of the vehicle.
- For an INS, this is typically the center of the IMU and for GPS it is the phase center of the antenna\(^2\).

### 1.2.4 The Sensor Head Frame

- Positioned at a convenient point on a sensor head such as:
  - the intersection of rotary axes
  - center of mounting plate
  - optical center of the hosted sensor

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1. The geographic pole is determined by the earth’s spin axis, not the magnetic field.
2. The antenna may be nowhere near the GPS receiver.
• At times, a rigid sensor head can and should be defined which tilts the body axes into coincidence with those of the sensor.

1.2.5 The Sensor Frame

• For video cameras, it is positioned on the optical axis at the center of projection behind the lens or on the image plane.
• For stereo systems, it is positioned either between both cameras or is associated with the center of projection of the image plane of one of them.
• For imaging laser rangefinders, it is positioned as the average point of convergence of the rays through each pixel.

1.2.6 The Wheel Frame

• This frame is positioned at the center of the wheel, on the axle.

1.3 The RPY Transform

• It is usually most convenient to express vehicle attitude in terms of three special angles called roll, pitch, and yaw.
• Luckily, most pan/tilt mechanisms are kinematically formed from a yaw rotation followed by a pitch with no roll, so they are a degenerate form of the above, more general, transform.
• A general homogeneous transform, called the RPY transform, can be formed which is similar in principle to the DH matrix, except that it has three rotations, and which can serve to transform between the body frame and all others.
• There are six degrees of freedom involved, three translations and three rotations, and each can be either a parameter or a variable.
• Let two general frames be defined as ‘a’ and ‘b’ and consider the moving axis operations which transform frame ‘a’ into coincidence with frame ‘b’. In order, these are:
  • translate along the (x,y,z) axes of frame ‘a’ by (u,v,w) until its origin coincides with that of frame ‘b’
  • rotate about the new z axis by an angle \( \psi \) called \textit{yaw}
  • rotate about the new x axis by an angle \( \theta \) called \textit{pitch}
  • rotate about the new y axis by an angle \( \phi \) called \textit{roll}

• Angles are measured counterclockwise positive according to the right hand rule.

• These operations are indicated below for the case of transforming the navigation frame into the body frame.

• The forward kinematic transform that represents this sequence of operations is, according to our rules for forward kinematics:
The RPY Transform

This matrix has the following interpretations:

- It rotates and translates points through the operations listed, in the order listed, with respect to the axes of ‘a’.
- Its columns represent the axes and origin of frame ‘b’ expressed in frame ‘a’ coordinates.
- It converts coordinates from frame ‘b’ to frame ‘a’.

The matrix can be considered to be the conversion from a pose vector of the form

\[
\begin{bmatrix}
  x \\
  y \\
  z \\
  \theta \\
  \phi \\
  \psi \\
\end{bmatrix}
\]

to a coordinate frame.
1.4 Inverse Kinematics for the RPY Transform

- The inverse kinematic solution to the RPY transform has at least two uses:
  - it gives the angles to which to drive a sensor head, or a directional antenna given the direction cosines of the goal frame
  - it gives the attitude of the vehicle given the body frame axes, which often correspond to the local tangent plane to the terrain over which it moves
- This solution can be considered to be the procedure for extracting a pose from a coordinate frame.
- There are many different ways to get the solution from different elements of the RPY transform. The one used here is useful for modelling terrain following of a vehicle.
- Proceeding as for a mechanism, the elements of the transform are assumed to be known:

\[
T^a_b = \begin{bmatrix}
  r_{11} & r_{12} & r_{13} & p_x \\
  r_{21} & r_{22} & r_{23} & p_y \\
  r_{31} & r_{32} & r_{33} & p_z \\
  0 & 0 & 0 & 1
\end{bmatrix}
\]
• The premultiplication set of equations will be used. The first equation is:

\[ T^a_b = \text{Trans}(u, v, w)\text{Rotz}(\psi)\text{Rotx}(\theta)\text{Roty}(\phi) \]

\[
\begin{bmatrix}
    r_{11} & r_{12} & r_{13} & p_x \\
    r_{21} & r_{22} & r_{23} & p_y \\
    r_{31} & r_{32} & r_{33} & p_z \\
    0 & 0 & 0 & 1 \\
\end{bmatrix}
= \begin{bmatrix}
    (c\psi c\phi - s\psi s\theta s\phi) & -s\psi c\theta & (c\psi s\phi + s\psi s\theta c\phi) & u \\
    (s\psi c\phi + c\psi s\theta s\phi) & c\psi c\theta & (s\psi s\phi - c\psi s\theta c\phi) & v \\
    -c\theta s\phi & s\theta & c\theta c\phi & w \\
    0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

• The translational elements are trivial. From the (1,2) and (2,2) elements:

\[ \psi = \text{atan}2(r_{22}, -r_{12}) \]

• This implies that yaw can be determined from a vector which is known to be aligned with the body y axis. The second equation is:

\[ [\text{Trans}(u, v, w)]^{-1}T^a_b = \text{Rotz}(\psi)\text{Rotx}(\theta)\text{Roty}(\phi) \]

\[
\begin{bmatrix}
    r_{11} & r_{12} & r_{13} & 0 \\
    r_{21} & r_{22} & r_{23} & 0 \\
    r_{31} & r_{32} & r_{33} & 0 \\
    0 & 0 & 0 & 1 \\
\end{bmatrix}
= \begin{bmatrix}
    (c\psi c\phi - s\psi s\theta s\phi) & -s\psi c\theta & (c\psi s\phi + s\psi s\theta c\phi) & 0 \\
    (s\psi c\phi + c\psi s\theta s\phi) & c\psi c\theta & (s\psi s\phi - c\psi s\theta c\phi) & 0 \\
    -c\theta s\phi & s\theta & c\theta c\phi & 0 \\
    0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

• which generates nothing new. The next equation is:

• From the (3,3) and (3,2) elements:

\[ \theta = \text{atan}2(r_{32}, -r_{12}s\psi + r_{22}c\psi) \]
Which implies that pitch can also be determined from a vector known to be aligned with the body y axis. A good solution for \( \phi \) is available from the (1,1) and (1,3) elements. However, for reasons of convenience, the solution will be delayed until the next equation. The next equation is:

\[
\phi = \text{atan2}(s\theta[-r_{11}s\psi + r_{21}c\psi] - r_{31}c\theta, (r_{11}c\psi + r_{21}s\psi))
\]

This implies that roll can be derived from a vector known to be aligned with the body x axis.
\[ [\text{Rot}_x(\theta)]^{-1} [\text{Rot}_z(\psi)]^{-1} [\text{Trans}(u, v, w)]^{-1} T^a_b = \text{Rot}_y(\phi) \]

\[
\begin{bmatrix}
(r_{11} c\psi + r_{21} s\psi) & (r_{12} c\psi + r_{22} s\psi) \\
-c\theta[-r_{11} s\psi + r_{21} c\psi] + r_{31} s\theta & c\theta[-r_{12} s\psi + r_{22} c\psi] + r_{32} s\theta \\
-s\theta[-r_{11} s\psi + r_{21} c\psi] + r_{31} c\theta & -s\theta[-r_{12} s\psi + r_{22} c\psi] + r_{32} c\theta \\
0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
(r_{13} c\psi + r_{23} s\psi) \\
c\theta[-r_{13} s\psi + r_{23} c\psi] + r_{33} s\theta \\
-s\theta[-r_{13} s\psi + r_{23} c\psi] + r_{33} c\theta \\
0
\end{bmatrix}
= \begin{bmatrix}
c\phi & 0 & s\phi & 0 \\
0 & 1 & 0 & 0 \\
-s\phi & 0 & c\phi & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
2 Angular Velocity

• The roll, pitch, and yaw angles are, as we have defined them, measured about moving axes. Therefore, they are a sequence of Euler angles, specifically, the z-x-y sequence\(^1\).

• The Euler angle definition of vehicle attitude has the disadvantage that the roll, pitch, and yaw angles are not the quantities that are actually indicated by strapped-down vehicle-mounted sensors such as gyros.

• The relationship between the rates of the Euler angles and the angular velocity vector is nonlinear. The angles are measured neither about the body axes nor about the navigation frame axes.

• The total angular velocity is the sum of three components, each measured about one of the intermediate axes in the chain of rotations which bring the navigation frame into coincidence with the body frame.

\[
\bar{\omega}^b = \begin{bmatrix} 0 \\ \dot{\phi} \\ 0 \end{bmatrix} + \text{rot}(y, \phi) \begin{bmatrix} \dot{\theta} \\ 0 \\ 0 \end{bmatrix} + \text{rot}(y, \phi)\text{rot}(x, \theta) \begin{bmatrix} 0 \\ 0 \\ \psi \end{bmatrix}
\]

\[
\bar{\omega}^b = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} c\phi \dot{\theta} - s\phi c\theta \dot{\psi} \\ \dot{\phi} + s\theta \dot{\psi} \\ s\phi \dot{\theta} + c\phi c\theta \dot{\psi} \end{bmatrix} = \begin{bmatrix} c\phi & 0 & -s\phi c\theta \\ 0 & 1 & s\theta \\ s\phi & 0 & c\phi c\theta \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{\phi} \\ \dot{\psi} \end{bmatrix}
\]

1. The sequence depends on the convention for assigning the directions of the linear axes.
• This result gives the vehicle angular velocity expressed in the body frame in terms of the Euler angle rates. Notice that when the vehicle is level, the x and y components are zero and the z component is just the yaw rate as expected.

• This relationship is also very useful in its inverted form. One can verify by substitution that:

\[
\begin{bmatrix}
\dot{\theta} \\
\dot{\phi} \\
\dot{\psi}
\end{bmatrix} = \begin{bmatrix}
\omega_x c\phi + \omega_z s\phi \\
\omega_y - t\theta [\omega_z c\phi - \omega_x s\phi] \\
[\omega_z c\phi - \omega_x s\phi] / c\theta
\end{bmatrix} = \begin{bmatrix}
c\phi & 0 & s\phi \\
t\theta s\phi & 1 & -t\theta c\phi \\
-s\phi & 0 & c\phi \\
-c\theta & 0 & c\theta
\end{bmatrix} \begin{bmatrix}
\omega_x \\
\omega_y \\
\omega_z
\end{bmatrix}
\]

because \( \omega_z c\phi - \omega_x s\phi = c\theta \psi \)
3 Actuator Kinematics

• For wheeled vehicles, the transformation from the angles of the steerable wheels and their velocities onto path curvatures, or equivalently angular velocities, can be very complicated.

• There can be more degrees of freedom of steer than are necessary. In this case, the equations which relate curvature to steer angle are **overdetermined**.

3.1 The Bicycle Model of Ackerman Steer Vehicles

• In one particular case, however, the steering mechanism is designed such that this will not be the case. This mechanism is used on most conventional automobiles and is called **Ackerman steering**.

• It is useful to approximate the kinematics of the Ackerman steering mechanism by assuming that the two front wheels turn slightly differentially so that the instantaneous center of rotation of the vehicle can be determined purely by kinematic means.

• Let the angular velocity vector directed along the body z axis be called $\dot{\beta}$. Using the bicycle model approximation, the path curvature $\kappa$, radius of curvature $R$, and steer angle $\alpha$ are related by the wheelbase $L$.

$$\frac{1}{R} = \frac{\tan \alpha}{L} = \frac{d\beta}{ds}$$
• Rotation rate is obtained from the speed $V$ as:

$$\dot{\beta} = \frac{d\beta}{ds} \frac{ds}{dt} = \kappa V = \frac{Vt\alpha}{L}$$

• Thus, the steer angle $\alpha$ is an indirect measurement of the ratio of $\dot{\beta}$ to velocity through the function:

$$\alpha = \tan\left(\frac{L\dot{\beta}}{V}\right) = \tan(\kappa L)$$

1. Curvature is $d\beta/ds$. Dividing top and bottom by $dt$, it's clear that curvature always measures the ratio of linear and angular velocities.
4 Kinematics of Video Cameras

• Many sensors used on robot vehicles are of the imaging variety. For this class of sensors, the process of image formation must be modelled.

• Typically, these transformations are not linear, and hence they cannot all be modelled by homogeneous transforms. This section provides the homogeneous transforms and nonlinear equations necessary for modelling such sensors.

4.1 Perspective Projection

• In the case of passive imaging systems, a system of lenses forms an image on an array of sensitive elements called a CCD.

• These systems include traditional video cameras and infra red cameras.

• The transformation from the sensor frame to the image plane row and column coordinates is the standard perspective projection.

• This type of transform is unique in two ways:
  • it reduces the dimension of the input vector by one and hence it discards information
  • it requires a post normalization step where the output is divided by the scale factor in order to re-establish a unity scale factor
• This transformation can be derived by similar triangles.

\[
\begin{align*}
x_i &= \frac{x_s f}{y_s + f} = \frac{x_s}{1 + y_s / f} \\
z_i &= \frac{z_s f}{y_s + f} = \frac{z_s}{1 + y_s / f} \\
y_i &= 0
\end{align*}
\]

\[
\begin{bmatrix}
x_i \\
y_i \\
z_i \\
w_i
\end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{1}{f} & 0 & 1 \end{bmatrix} \begin{bmatrix} x_s \\ y_s \\ z_s \\ 1 \end{bmatrix}
\]

\[
P_s^i = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{1}{f} & 0 & 1 \end{bmatrix}
\]

• Note in particular that all projections are not invertible. Here, the second row is all zeros, so the matrix is singular.

• This, of course, is the ultimate source of the difficulty of measuring scene geometry with a single camera. For this reason, this transform is identified by the special capital letter P.
5 Kinematics of Laser Rangefinders

• Kinematics of reflecting a laser beam from a mirror are central to the operation of the current generation of laser rangefinders.

• There are at least two ways to go to model them:
  • The reflection operator
  • The “mechanism” forward kinematics where we consider the operations on the laser beam to be the mechanism.

• Always be careful to account for mirror gain. This is the amount by which the rate of rotation of the laser beam is different from the rate of rotation of the mirror.

5.1 The Reflection Operator

• This can be used when the translations of the laser beam can be ignored.

• From Snell’s law for reflection of a ray:
  • The incident ray, the normal to the surface, and the reflected ray, all lie in the same plane.
  • The angle of incidence equals the angle of reflection.

• From these two rules, a very useful matrix operator can be formulated to reflect a vector off of any surface, given the unit normal to the surface.

• Consider a vector \( \hat{\mathbf{v}}_i \), not necessarily a unit vector, which impinges on a reflecting surface at a point where the unit normal to the surface is \( \hat{n} \). Unless they are parallel, the incident and normal vector define a plane, which will be called the reflection plane.
- Drawing both vectors in this plane, it is clear that Snell’s law can be written in many forms:

\[ \hat{v}_r = \hat{v}_i - 2(\hat{v}_i \cdot \hat{n})\hat{n} \]

\[ \hat{v}_r = \hat{v}_i - 2\hat{v}_i \cos \theta \hat{n} \]

\[ \hat{v}_r = \hat{v}_i - 2(\hat{n} \otimes \hat{n})\hat{v}_i \]

\[ \hat{v}_r = \text{Ref}(\hat{n})\hat{v}_i \]

Where the outer product (\( \otimes \)) of the normal with itself was used in forming the matrix equivalent.

- The result is expressed in the same coordinates in which both the normal and the incident ray were expressed.

- Notice that a reflection is equivalent to a rotation of twice the angle of incidence about the normal to the reflection plane.

- A similar matrix refraction operator can be defined.

- In order to model rangefinders, the laser beam will be modelled by a unit vector since the length of the beam is immaterial. The unit vector is operated upon by the reflection operator - one reflection for each mirror. The ultimate result of all reflections will be expressed in the original coordinate system.
The results of such an analysis give the orientation of the laser beam as a function of the actuated mirror angles, but it says nothing about where the beam is positioned in space. The precise position of the beam is not difficult to calculate and is important in the sizing of mirrors. From the point of view of computing kinematics, beam position can often be ignored.

5.2 Kinematics of the Azimuth Scanner

- The azimuth scanner is a generic name for a class of laser rangefinders with equivalent kinematics. In this scanner, the laser beam undergoes the azimuth rotation/reflection first and the elevation rotation/reflection second.

- Examples are the ERIM and Perceptron. Both scanners are 2D scanning laser rangefinders employing a “polygonal” azimuth mirror and a flat “nodding” elevation mirror. The mirrors move as shown below:
5.2.1 Forward Kinematics

- A coordinate system called the “s” system is fixed to the sensor with y pointing out the front of the sensor and x pointing out the right side. The beam enters along the $x_s$ axis. It reflects off the polygonal mirror which rotates about the $y_s$ axis. It then reflects off the nodding mirror, to leave the housing roughly aligned with the $y_s$ axis.

- First, the beam is reflected from the laser diode about the normal to the polygonal mirror. Computation of the output of the polygonal mirror can be done by inspection - noting that the beam rotates by twice the angle of the mirror because it is a reflection operation. The z-x plane contains both the incident and normal vectors. The datum position of the mirror should correspond to a perfectly vertical beam, so the datum for the mirror rotation angle is chosen appropriately. Consider an input beam $\hat{v}_m$ along the $x_s$ axis and reflect it about the mirror by inspection:

- Notice that this vector is contained within the $x_s$-$z_s$ plane. Now this result must be reflected about the nodding mirror. Notice that, at this point, $\hat{v}_p$ cannot be simply rotated around the x axis since the axis of rotation which is equivalent to a
reflection is normal to both \( \hat{v}_p \) and \( \hat{n}_n \). Since \( \hat{v}_p \) is not always in the \( y_s-z_s \) plane, the \( x_s \) axis is not always the axis of rotation.

\[
\hat{v}_p = [\sin{\psi} \ 0 \ \cos{\psi}]^T
\]

\[
\text{put } \frac{\alpha}{2} = \frac{\pi}{4} - \frac{\theta}{2}
\]

\[
\hat{n}_n = \begin{bmatrix} 0 & \frac{\alpha}{2} & -\frac{\alpha}{2} \end{bmatrix}^T
\]

\[
\hat{v}_n = \text{Ref}(\hat{n}_n)\hat{v}_p = \hat{v}_p - 2(\hat{v}_p \cdot \hat{n}_n)\hat{n}_n
\]

\[
\hat{v}_n = \begin{bmatrix} \sin{\psi} \\ 0 \\ \cos{\psi} \end{bmatrix} + 2\cos{\psi}\frac{\alpha}{2} \begin{bmatrix} 0 \\ \frac{\alpha}{2} \\ -\frac{\alpha}{2} \end{bmatrix} = \begin{bmatrix} \sin{\psi} \\ \cos{\psi}\frac{\pi}{2} - \theta \\ -\cos{\psi}\frac{\pi}{2} - \theta \end{bmatrix}
\]

\[
\hat{v}_n = \begin{bmatrix} \sin{\psi} [\cos{\psi}\theta] - [\cos{\psi}\theta] \end{bmatrix}^T
\]

- This result is summarized in the following figure:

\[
v_s = \begin{bmatrix} x_s \\ y_s \\ z_s \end{bmatrix} = \begin{bmatrix} R_s \psi \\ R \theta \cos{\psi} \\ -R_s \theta \cos{\psi} \end{bmatrix}
\]

- Thus, the kinematics of the azimuth scanner are equivalent to a rotation around the \( x_s \) axis followed by a rotation around the \textit{new} \( z_s \) axis. This is also equivalent to two rotations in the opposite order about fixed axes.
5.2.2 Forward Imaging Jacobian

- The imaging Jacobian provides the relationship between the differential quantities in the sensor frame and the associated position change in the image. The Jacobian is:

\[
\begin{bmatrix}
R_x \\
\psi \\
\theta
\end{bmatrix}
\begin{bmatrix}
x_s \\
y_s \\
z_s
\end{bmatrix}
= \begin{bmatrix}
R \psi \\
R \theta c \psi \\
-R \theta c \psi
\end{bmatrix}
\]

\[
J_i^s = \frac{\partial v_s}{\partial v_s} = \begin{bmatrix}
dx_s & dx_s & dx_s \\
\frac{\partial R}{\partial \psi} & \frac{\partial R}{\partial \theta} & \frac{\partial R}{\partial \psi} \\
\frac{\partial y_s}{\partial \psi} & \frac{\partial y_s}{\partial \theta} & \frac{\partial y_s}{\partial \psi} \\
\frac{\partial z_s}{\partial \psi} & \frac{\partial z_s}{\partial \theta} & \frac{\partial z_s}{\partial \psi}
\end{bmatrix}
= \begin{bmatrix}
s \psi & R \chi \psi & 0 \\
c \theta c \psi & -R \theta c \psi & -R \theta c \psi \\
-s \theta c \psi & R \theta s \psi & -R \theta c \psi
\end{bmatrix}
\]

5.2.3 Inverse Kinematics

- The forward transform is easily inverted.

\[
\begin{bmatrix}
R \\
\psi \\
\theta
\end{bmatrix}
= \begin{bmatrix}
\sqrt{x_s^2 + y_s^2 + z_s^2} \\
\arctan\left(\frac{x_s}{\sqrt{y_s^2 + z_s^2}}\right) \\
\arctan\left(\frac{-z_s}{y_s}\right)
\end{bmatrix}
= h(x_s, y_s, z_s)
\]
5.2.4 Inverse Imaging Jacobian

- The imaging Jacobian provides the relationship between the differential quantities in the sensor frame and the associated position change in the image. The Jacobian is:

\[
\mathbf{v}_i = \begin{bmatrix} R \\ \psi \\ \theta \end{bmatrix} = \begin{bmatrix} \sqrt{x_s^2 + y_s^2 + z_s^2} \\ \arctan(x_s / (\sqrt{y_s^2 + z_s^2})) \\ \arctan(-z_s / y_s) \end{bmatrix} \quad \mathbf{v}_s = \begin{bmatrix} x_s \\ y_s \\ z_s \end{bmatrix}
\]

\[
J^i_s = \frac{\partial \mathbf{v}^i}{\partial \mathbf{v}^s} = \begin{bmatrix} \frac{\partial x_s}{\partial x_s} & \frac{\partial x_s}{\partial y_s} & \frac{\partial x_s}{\partial z_s} \\ \frac{\partial y_s}{\partial x_s} & \frac{\partial y_s}{\partial y_s} & \frac{\partial y_s}{\partial z_s} \\ \frac{\partial z_s}{\partial x_s} & \frac{\partial z_s}{\partial y_s} & \frac{\partial z_s}{\partial z_s} \end{bmatrix} = \begin{bmatrix} \frac{x_s}{R} & \frac{y_s}{R} & \frac{z_s}{R} \\ \frac{\sqrt{y_s^2 + z_s^2}}{R^2} & \frac{y_s}{\sqrt{y_s^2 + z_s^2}} & \frac{z_s}{\sqrt{y_s^2 + z_s^2}} \\ 0 & \frac{z_s}{y_s^2 + z_s^2} & \frac{-y_s}{y_s^2 + z_s^2} \end{bmatrix}
\]

5.2.5 Analytic Range Image of Flat Terrain

- Given the basic kinematic transform, many analyses can be performed. The first is to compute an analytic expression for
the range image of a perfectly flat piece of terrain. Let the sensor fixed “s” coordinate system be mounted at a height $h$ and tilted forward by a tilt angle of $\beta$. Then, the transform from sensor coordinates to global coordinates is:

$$
\begin{align*}
    x_g &= x_s \\
    y_g &= y_s c\beta + z_s s\beta \\
    z_g &= -y_s s\beta + z_s c\beta + h
\end{align*}
$$

- If the kinematics are substituted into this, the transform from the polar sensor coordinates to global coordinates is obtained:

$$
\begin{align*}
    x_g &= R_s \psi \\
    y_g &= (R_c \theta c \psi) c\beta - (R_s \theta c \psi) s\beta = R_c \theta b c \psi \\
    z_g &= (-R_c \theta c \psi) s\beta - (R_s \theta c \psi) c\beta + h = h - R_s \theta b c \psi
\end{align*}
$$

- Now by setting $z_g = 0$ and solving for $R$, the expression for $R$ as a function of the beam angles $\psi$ and $\theta$ for flat terrain is obtained. This is an analytic expression for the range image of flat terrain under the azimuth transform.

$$
R = h/(c\psi s\beta)$$
• Notice that when $R$ is large $s\theta \beta = h/R$. As a check on the range image formula, the resulting range image is shown below for $h = 2.5$, $\beta = 16.5^\circ$, a horizontal field of view of 140°, a vertical field of view of 30°, and an IFOV of 5 mrads. It has 490 columns and 105 rows. The edges correspond to contours of constant range of 20 meters, 40 meters, 60 meters, etc.

• The curvature of the contours of range is intrinsic to the sensor kinematics and is independent of the sensor tilt. Substituting this back into the coordinate transform, the coordinates where each ray intersects the groundplane are:

\[
\frac{x_g}{h \psi / s \theta \beta} = \frac{y_g}{h / t \theta \beta} = \frac{z_g}{0}
\]

• Notice that the y coordinate is independent of $\psi$ and hence, lines of constant elevation in the image are \textit{straight lines along the y-axis} on flat terrain.

• From the previous result, it can be verified by substitution and some algebra that:

\[
\left[\frac{x_g}{h \psi / s \theta \beta}\right]^2 - y_g^2 = h^2
\]

• Thus lines of constant azimuth are \textit{hyperbolas} on the groundplane.
5.2.6 Resolution

- The Jacobian of the groundplane transform has many uses. Most important of all, it provides a measure of sensor resolution on the ground plane. Differentiating the previous result:

\[
\begin{bmatrix}
dx_g \\
dy_g 
\end{bmatrix} = \begin{bmatrix}
\frac{h(\sec \psi)^2}{s \theta \beta} & -ht \psi c \theta \beta \\
0 & \frac{h}{(s \theta \beta)^2}
\end{bmatrix} \times \begin{bmatrix}
d\psi \\
d\theta
\end{bmatrix}
\]

- The determinant of the Jacobian relates differential areas:

\[
dx_g dy_g = \left[ \frac{(h \sec \psi)^2}{(s \theta \beta)^3} \right] d\psi d\theta
\]

- Notice that when \( R \) is large and \( \beta = \beta \), the Jacobian norm can be approximated by:

\[
\left[ \frac{(h \sec \psi)^2}{(s \theta \beta)^3} \right] \approx \frac{h^2}{(h \beta)^3} = R^2 \left( \frac{h}{R} \right)
\]

- Thus, the pixel density on the ground is proportional to the cube of the range.
5.2.7 Azimuth Scanning Pattern

- The scanning pattern is shown in the following figure with a 10 m grid superimposed for reference purposes. Only every fifth pixel is shown in azimuth to avoid clutter.

5.3 Simple Scanner Kinematics

- The simplest possible 2D scanner can be constructed by mounting a 1D scanner on a pan table.
- Although its simple mechanically, its a tough math model relative to others.
• We will model this one with our fundamental operators and rules for forward kinematic modelling.

5.3.1 Forward Kinematics

• The homogeneous coordinates of a range pixel in the mirror (m) frame are:

\[
\mathbf{v}_m = \begin{bmatrix} 0 & 0 & -R & 1 \end{bmatrix}^T
\]

• To convert the coordinates of this pixel to the intermediate (i) frame, we must generate the relevant transform as follows. The operations which bring frame i into coincidence with frame m are:

- Translate along \( y_i = y_m \) a distance \( l_e \).
- Rotate around \( y_i = y_m \) an angle \( \theta \).
- Hence, the matrix \( T_m^i \), which converts coordinates of a point from frame m to frame i is:
\[ T_m^i = Trans(0, l_e, 0)Roty(\theta) \]

\[
T^i_m = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & l_e \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
T^i_m = \begin{bmatrix}
c\theta & 0 & s\theta & 0 \\
0 & 1 & 0 & l_e \\
s\theta & 0 & c\theta & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
T^m_i = \begin{bmatrix}
c\theta & 0 & -s\theta & 0 \\
0 & 1 & 0 & -l_e \\
s\theta & 0 & c\theta & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

• Therefore, the coordinates of a range pixel in the i frame are:

\[
v_i = T^i_m v_m
\]

\[
\begin{bmatrix}
x \\
y \\
z \\
w_i
\end{bmatrix} =
\begin{bmatrix}
c\theta & 0 & s\theta & 0 \\
0 & 1 & 0 & l_e \\
s\theta & 0 & c\theta & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
-R \\
1
\end{bmatrix}
\]

\[
v_i = \begin{bmatrix}
-Rs\theta & l_e & -Rc\theta & 1
\end{bmatrix}^T
\]

• To convert the coordinates of this pixel to the sensor (s) frame, we must generate the relevant transform as follows. The operations which bring frame s into coincidence with frame i are:

  • Translate along \( z_s = z_i \) a distance \( l_a \).

  • Rotate around \( z_s = z_i \) an angle \( \psi \).

  • Hence, the matrix \( T_i^s \) which converts coordinates of a point from frame i to frame s is:
\[ T_i^s = \text{Trans}(0, 0, l_a)\text{Rotz}(\psi) \]

\[
T_i^s = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix} \begin{bmatrix}
c\psi & s\psi & 0 \\
s\psi & c\psi & 0 \\
0 & 0 & 1 \\
\end{bmatrix} \begin{bmatrix}
l_a & 0 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

\[
T_s^i = \begin{bmatrix}
c\psi & -s\psi & 0 \\
s\psi & c\psi & 0 \\
0 & 0 & 1 \\
\end{bmatrix} \begin{bmatrix}
c\psi & s\psi & 0 \\
-s\psi & c\psi & 0 \\
0 & 0 & 1 \\
\end{bmatrix} \begin{bmatrix}
l_a & 0 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

Therefore, the coordinates of a range pixel in the s frame are:

\[
v_s = T_i^s v_i
\]

\[
\begin{bmatrix}
x \\
y \\
z \\
w
\end{bmatrix} = \begin{bmatrix}
c\psi & -s\psi & 0 & 0 \\
s\psi & c\psi & 0 & 0 \\
0 & 0 & 1 & l_a \\
0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
-R_s\theta \\
l_e \\
-R_c\theta \\
1
\end{bmatrix}
\]

\[
v_s = \begin{bmatrix}
-R_c\psi s\theta - l_e s\psi \\
-R_s\psi s\theta + l_e c\psi \\
-R_c\theta + l_a \\
1
\end{bmatrix}
\]

The last expression is the complete sensor forward kinematics solution which converts coordinates from range, azimuth, elevation to a cartesian (x,y,z) position with respect to the sensor frame.
5.3.2 Inverse Kinematics

- The goal of the inverse kinematics is to compute the range, and the azimuth and elevation angles which correspond to a given cartesian (x,y,z) position expressed with respect to the sensor frame.

- The forward kinematic relationship can be rewritten as follows:

\[
\begin{align*}
\bar{v}_i &= T_m^i v_m \\
\bar{v}_s &= T_s^i \bar{v}_i \\
\therefore \bar{v}_s &= T_s^i (T_m^i v_m)
\end{align*}
\]

- The inverse kinematic solution can be obtained by assuming that \( v_s \) is known and solving for the range, azimuth, and elevation of the point. Premultiplying the above by \( T_s^i \):

\[
T_s^i \bar{v}_s = T_m^i v_m \quad (= \bar{v}_i)
\]

\[
\begin{bmatrix}
\cos \psi & \sin \psi & 0 & 0 \\
-\sin \psi & \cos \psi & 0 & 0 \\
0 & 0 & 1 & -l_a \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
w_s
\end{bmatrix} =
\begin{bmatrix}
\cos \theta & 0 & \sin \theta & 0 \\
0 & 1 & 0 & l_e \\
-\sin \theta & 0 & \cos \theta & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
-R \\
1
\end{bmatrix}
\]
This generates the following three equations:

\[ \begin{align*}
1) \ c \psi x_s + s \psi y_s &= -Rs \theta \\
2) \ -s \psi x_s + c \psi y_s &= l_e \\
3) \ z_s - l_a &= -Rc \theta
\end{align*} \]

- First, we solve 2) directly for the azimuth angle. Then, we solve 1) and 3) for the elevation angle. Finally, 3) can be solved for the range when the elevation angle is known. The result is:

\[
\begin{align*}
\psi &= \text{atan2}(-y_s, -x_s) - \text{atan2}(l_e, \pm \sqrt{x_s^2 + y_s^2 - l_e^2}) \\
\theta &= \text{atan2}(-c \psi x_s - s \psi y_s, l_a - z_s) \\
R &= (l_a - z_s) / (c \theta)
\end{align*}
\]

- which is the inverse kinematics solution.
6 Summary

• The RPY matrix is yet another compound orthogonal operator matrix. Unlike the DH matrix, it has 6 dof so it is completely general.

• Angular velocity is related in a complicated manner to the rates of roll, pitch, and yaw angles.

• Video cameras are modelled by a perspective projection.

• Laser rangefinder models are nonlinear and cannot be represented by a constant homogeneous transform like a camera can.

• However, our mechanism modelling rules apply perfectly and one can also use a reflection operator to model them.