Lecture 14 - Model Predictive Control
Part 1: The Concept

- History and industrial application resource:
  - Joe Qin, survey of industrial MPC algorithms

- Emerging applications

- State-based MPC
  - Conceptual idea of MPC
  - Optimal control synthesis

- Example
  - Lateral control of a car

- Stability

- Lecture 15: Industrial MPC
MPC concept

• MPC = Model Predictive Control
• Also known as
  – DMC = Dynamical Matrix Control
  – GPC = Generalized Predictive Control
  – RHC = Receding Horizon Control
• Control algorithms based on
  – Numerically solving an optimization problem at each step
  – Constrained optimization – typically QP or LP
  – Receding horizon control
• More details need to be worked out for implementation
Receding Horizon Control

- Receding Horizon Control concept

- At each time step, compute control by solving an open-loop optimization problem for the prediction horizon
- Apply the first value of the computed control sequence
- At the next time step, get the system state and re-compute

Diagram:
- prediction horizon
- Plant
- Plant Model
- RHC
- future input trajectory
- predicted future output
- current dynamic system states
Current MPC Use

- Used in a majority of existing multivariable control applications
- Technology of choice for many new advanced multivariable control application
- Success rides on the computing power increase
- Has many important practical advantages
MPC Advantages

• Straightforward formulation, based on well understood concepts
• Explicitly handles constraints
• Explicit use of a model
• Well understood tuning parameters
  – Prediction horizon
  – Optimization problem setup
• Development time much shorter than for competing advanced control methods
• Easier to maintain: changing model or specs does not require complete redesign, sometimes can be done on the fly
History

• First practical application:
  – DMC – Dynamic Matrix Control, early 1970s at Shell Oil
  – Cutler later started Dynamic Matrix Control Corp.

• Many successful industrial applications

• Theory (stability proofs etc) lagging behind 10-20 years.

• See an excellent resource on industrial MPC
  – Joe Qin, Survey of industrial MPC algorithms
  – history and formulations
### Some Major Applications

<table>
<thead>
<tr>
<th>Area</th>
<th>DMC Corp.</th>
<th>Setpoint Inc.</th>
<th>Honeywell Profimatics</th>
<th>Adersa</th>
<th>Treiber Controls</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Refining</td>
<td>360</td>
<td>320</td>
<td>290</td>
<td>280</td>
<td>250</td>
<td>1500</td>
</tr>
<tr>
<td>Petrochemicals</td>
<td>210</td>
<td>40</td>
<td>40</td>
<td>-</td>
<td>-</td>
<td>290</td>
</tr>
<tr>
<td>Chemicals</td>
<td>10</td>
<td>20</td>
<td>10</td>
<td>3</td>
<td>150</td>
<td>193</td>
</tr>
<tr>
<td>Pulp and Paper</td>
<td>10</td>
<td>-</td>
<td>30</td>
<td>-</td>
<td>5</td>
<td>45</td>
</tr>
<tr>
<td>Gas</td>
<td>-</td>
<td>-</td>
<td>5</td>
<td>-</td>
<td>-</td>
<td>5</td>
</tr>
<tr>
<td>Utility</td>
<td>-</td>
<td>-</td>
<td>2</td>
<td>-</td>
<td>-</td>
<td>2</td>
</tr>
<tr>
<td>Air Separation</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>Mining/Metallurgy</td>
<td>-</td>
<td>2</td>
<td>-</td>
<td>7</td>
<td>6</td>
<td>15</td>
</tr>
<tr>
<td>Food Processing</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>41</td>
<td>-</td>
<td>41</td>
</tr>
<tr>
<td>Furnaces</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>42</td>
<td>-</td>
<td>42</td>
</tr>
<tr>
<td>Aerospace/Defense</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>13</td>
<td>-</td>
<td>13</td>
</tr>
<tr>
<td>Automotive</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>7</td>
<td>-</td>
<td>7</td>
</tr>
<tr>
<td>Other</td>
<td>10</td>
<td>20</td>
<td>-</td>
<td>45</td>
<td>-</td>
<td>75</td>
</tr>
<tr>
<td>Total</td>
<td>600</td>
<td>402</td>
<td>377</td>
<td>438</td>
<td>416</td>
<td>2293</td>
</tr>
<tr>
<td>Largest App</td>
<td>603x283</td>
<td>35x28</td>
<td>26x20</td>
<td>-</td>
<td>24x19</td>
<td></td>
</tr>
</tbody>
</table>

- 1995 data, probably 1-2 order of magnitude growth by now
Emerging MPC applications

- **Nonlinear MPC**
  - just need a computable model (simulation)
  - NLP optimization
- **Hybrid MPC**
  - discrete and parametric variables
  - combination of dynamics and discrete mode change
  - mixed-integer optimization (MILP, MIQP)
- **Engine control**
- **Large scale operation control problems**
  - Operations management (control of supply chain)
  - Campaign control
Emerging MPC applications

- Vehicle path planning and control
  - nonlinear vehicle models
  - world models
  - receding horizon preview
Emerging MPC applications

- Spacecraft rendezvous with space station
  - visibility cone constraint
  - fuel optimality

- Underwater vehicle guidance

- Missile guidance

From Richards & How, MIT
State-based control synthesis

• Consider single input system for better clarity
  \[ x(t + 1) = Ax(t) + Bu(t) \]
  \[ y(t) = Cx(t) \]

• Infinite horizon optimal control
  \[ \sum_{\tau=t+1}^{\infty} (y(\tau))^2 + r(u(\tau) - u(\tau - 1))^2 \rightarrow \min \]
  subject to: \[ |u(\tau)| \leq u_0 \]

• Solution = Optimal Control Synthesis
State-based MPC – concept

- Optimal control trajectories converge to (0,0)
- If $N$ is large, the part of the problem for $t > N$ can be neglected
- Infinite-horizon optimal control $\approx$ horizon-$N$ optimal control
State-based MPC

• Receding horizon control; $N$-step optimal

\[ J = \sum_{\tau=t+1}^{t+N} (y(\tau))^2 + r(u(\tau) - u(\tau - 1))^2 \rightarrow \min \]

subject to: \[ |u(\tau)| \leq u_0, \]
\[ x(t+1) = Ax(t) + Bu(t) \]
\[ y(t) = Cx(t) \]

• Solution ≈ Optimal Control Synthesis

\[ x(t) \rightarrow [\text{MPC Problem Solver}] \rightarrow u(t) \]
Predictive Model

- Predictive system model

\[ Y = Gx + HU + Fu \quad \text{initial condition response} + \text{control response} \]

Predicted output
\[
\begin{bmatrix}
y(t + 1) \\
\vdots \\
y(t + N)
\end{bmatrix}
\]

Future control input
\[
\begin{bmatrix}
u(t + 1) \\
\vdots \\
u(t + N)
\end{bmatrix}
\]

Current state (initial condition)
\[ x = x(t) \]

\[ u = u(t) \rightarrow \text{computed at the previous step} \]

- Model matrices

\[
G = \begin{bmatrix}
CA \\
\vdots \\
CA^n
\end{bmatrix} \\
H = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
CB & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
CA^{N-2}B & CA^{N-3}B & \cdots & 0
\end{bmatrix} = \begin{bmatrix}
h(1) & 0 & \cdots & 0 \\
h(2) & h(1) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
h(N) & h(N-1) & \cdots & h(1)
\end{bmatrix} \\
F = \begin{bmatrix}
h(2) \\
h(3) \\
\vdots \\
h(N+1)
\end{bmatrix}
\]
Computations Timeline

- Assume that control $u$ is applied and the state $x$ is sampled at the same instant $t$
- Entire sampling interval is available for computing $u$
MPC Optimization Problem Setup

• MPC optimization problem
\[ J = Y^T Y + rU^T D^T D U \rightarrow \text{min} \]
subject to: \( |U| \leq u_0 \),
\[ Y = Gx + HU + Fu \]

• This is a QP problem

• Solution
\[ x(t) \rightarrow [\text{MPC Problem, QP Solver}] \rightarrow U \rightarrow u(t + 1) = U(1) \]
QP solution

• QP Problem:
\[ AU \leq b \]
\[ J = \frac{1}{2} U^T QU + f^T U \rightarrow \min \]

\[ Q = r D^T D + H^T H \]
\[ f = H^T (Gx + Fu) \]

• Standard QP codes can be used
Linear MPC

- Nonlinearity is caused by the constraints
- If constraints are inactive, the QP problem solution is
  \[ U = Q^{-1}f \quad u = l^T U \]
- This is linear state feedback
  \[ u(t + 1) = l^T \left( rD^T D + H^T H \right)^{-1} H^T \left( Gx(t) + Fu(t) \right) \]
  \[ u = z^{-1} Kx + z^{-1} Su \]
  \[ K = l^T \left( rD^T D + H^T H \right)^{-1} H^T G \]
  \[ S = l^T \left( rD^T D + H^T H \right)^{-1} H^T F \]
- Can be analyzed as a linear system, e.g., check eigenvalues
  \[ u = \frac{z^{-1}}{1 - Sz^{-1}} Kx \]
  \[ zx = Ax + Bu \]
Nonlinear MPC Stability

• **Theorem** - from Bemporad et al (1994)

  Consider a MPC algorithm for a linear plan with constraints. Assume that there is a terminal constraint
  \[
  x(t + N) = 0 \quad \text{for predicted state } x \text{ and} \\
  u(t + N) = 0 \quad \text{for computed future control } u
  \]
  If the optimization problem is feasible at time \( t \), then the coordinate origin is stable.

**Proof.**

  Use the performance index \( J \) as a Lyapunov function. It decreases along the finite feasible trajectory computed at time \( t \). This trajectory is suboptimal for the MPC algorithm, hence \( J \) decreases even faster.
MPC Stability

• The analysis could be useful in practice
  – Theory says a terminal constraint is good
• MPC stability formulations
  (Mayne et al, *Automatica*, 2000)
• Terminal equality constraint
• Terminal cost function
  – Dual mode control – infinite horizon
• Terminal constraint set
  – Increase feasibility region
• Terminal cost and constraint set
Example: Lateral Control of a Car

- Preview Control – MacAdam’s driver model (1980)
- Consider predictive control design
- Simple kinematical model of a car driving at speed $V$

\[
\begin{align*}
\dot{x} &= V \cos a \\
\dot{y} &= V \sin a \\
\dot{a} &= u
\end{align*}
\]

lateral displacement
steering
Lateral Control of a Car - Model

- Assume a straight lane – tracking a straight line
- Linearized system: assume $a \ll 1$
  \[ \sin a \approx a \quad \dot{y} = Va \]
  \[ \cos a \approx 1 \quad \dot{a} = u \]
- Sampled-time equations (sampling time $T_s$)
  \[ a(t + 1) = a(t) + u(t)T_s \]
  \[ y(t + 1) = y(t) + a(t)VT_s + u(t) \cdot 0.5VT_s^2 \]
Lateral Control of a Car - MPC

State-space system: \( x(t + 1) = Ax(t) + Bu(t) \)

\[
x(t) = \begin{bmatrix} a(t) \\ y(t) \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 \\ VT_s & 1 \end{bmatrix}, \quad B = \begin{bmatrix} T_s \\ 0.5VT_s^2 \end{bmatrix}
\]

Observation: \( y(t) = Cx(t) \)

\[
C = \begin{bmatrix} 0 & 1 \end{bmatrix}
\]

• Formulate predictive model

\[
Y = Gx + HU + Fu
\]

• MPC optimization problem

\[
J = (Gx + HU + Fu)^T (Gx + HU + Fu) + rU^T D^T DU \rightarrow \min
\]

subject to: \( U \leq u_0 \),

• Solution: \( x(t) \rightarrow [\text{MPC QP}] \rightarrow U \rightarrow u(t + 1) = U(1) \)
Impulse Responses

IMPULSE RESPONSE FOR LATERAL ERROR

IMPULSE RESPONSE FOR HEADING ANGLE
Lateral Control of a Car - Simulation

Simulation Results:

- \( V = 50 \) mph
- Sample time of 200ms
- \( N = 20 \)
- All variables in SI units
- \( r = 1 \)
Control Design Issues

• Several important issues remain
  – They are not visible in this simulation
  – Will be discussed in Lecture 15 (MPC, Part 2)
• All states might not be available
• Steady state error
  – Need integrator feedback
• Large angle deviation
  – Linearized model deficiency
  – Introduce soft constraint