On the Number of Synchronizing Colorings of Digraphs

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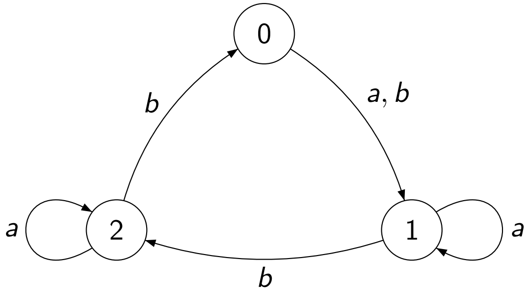
20 August, 2015

We consider deterministic finite automata: $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$
$|Q| = n$ and $|\Sigma| = k$

$\mathscr{A}$ is called synchronizing if there exists a word $w \in \Sigma^*$ whose action resets $\mathscr{A}$, that is, leaves the automaton in one particular state no matter which state in Q it started at: $\delta(q, w) = \delta(q', w)$ for all $q, q' \in Q$

Any w with this property is a synchronizing word for $\mathscr{A}$

The minimum length of synchronizing words for $\mathscr{A}$ is called the reset threshold of $\mathscr{A}$ and denoted by $\mathrm{rt}(\mathscr{A})$



The word *abba* is synchronizing

It resets the automaton to the state 1

Theorem (Černý, 1964)

For every positive integer n, there is an automaton $\mathscr{C}_n$ with n states and the reset threshold equal to $(n-1)^2$.

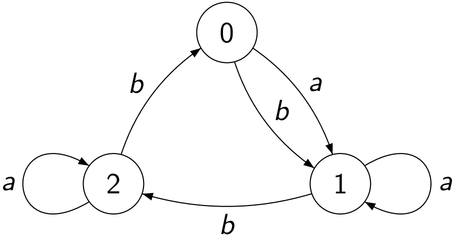
Conjecture (Černý, 1969)

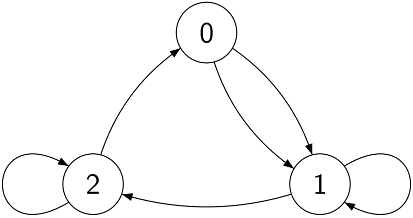
The reset threshold of a synchronizing automaton with n states is at most $(n-1)^2$.

Widely open: best upper bound is cubic (Pin, 1983).

Reformulation : Given a set S of mappings from n-element set into itself. If a constant mapping can be represented as a product of mappings from S, then there exists such a product of length $(n-1)^2$.

Automaton and its underlying digraph





Digraph and its coloring

We consider only k-out-regular digraphs!

What are the properties of the digraph $\mathcal{G}$ of a synchronizing automaton $\mathscr{A}$?

Lemma (folklore)

If $\mathscr{A}$ is strongly connected then $\mathcal{G}$ is primitive, i.e. greatest common divisor of the lengths of cycles is equal to 1.

Otherwise, $\mathcal{G}$ is a cyclic multi-partite digraph.

Lemma (folklore)

If $\mathscr{A}$ is not strongly connected then $\mathcal{G}$ has a unique primitive strongly connected sink component.

Are these conditions sufficient?

The Road Coloring Problem (Ader, Goodwyn and Weiss, 1977)

Does every primitive strongly connected digraph have a synchronizing coloring?

Theorem (Trahtman, 2007)

Every primitive strongly connected digraph has a synchronizing coloring.

After a crucial observation by Culik II, Karhumäki and Kari in 2001.

Theorem (Béal, Perrin, 2008)

There is an $O(kn^2)$-time algorithm to find a synchronizing coloring, where n is the number of states and k is the number of letters.

Are non-synchronizing colorings more common than synchronizing?

The synchronizing ratio of a digraph $\mathcal{G}$ is

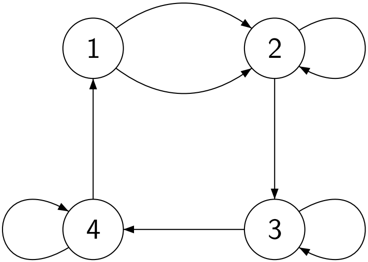
$$\mathrm{SynRatio}(\mathcal{G}) = \frac{\text{the number of synchronizing colorings of } \mathcal{G}}{\text{the number of all possible colorings of } \mathcal{G}}$$

Note : there are always $(k!)^n$ colorings

Proposition

If a digraph $\mathcal{G}$ have at least one synchronizing coloring then the synchronizing ratio of $\mathcal{G}$ is equal to the synchronizing ratio of it's unique strongly connected sink component.

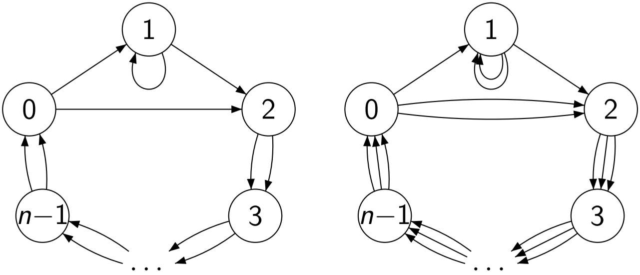
We are interested in worst and average case behaviour of $SynRatio(\mathcal{G})$.



Theorem (folklore, the proof is in our paper)

The synchronizing ratio of the underlying digraph of the Černý's automaton is equal to 1.

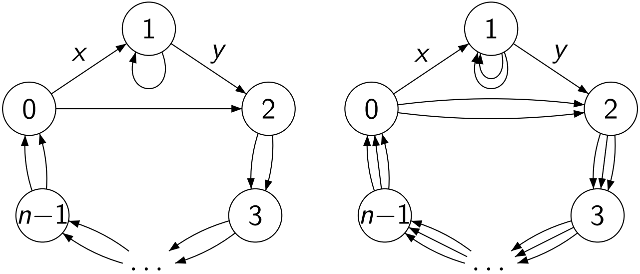
The digraphs $\mathcal{G}_{n,2}$ and $\mathcal{G}_{n,3}$



Theorem

For every $n > 3$ there is a k-out-regular digraph $\mathcal{G}_{n,k}$ with n vertices and the synchronizing ratio $1 - \frac{1}{k}$.

The digraphs $\mathcal{G}_{n,2}$ and $\mathcal{G}_{n,3}$



If $x = y$ then all letters are permutations, a non-synchronizing coloring.

If $x \neq y$ then x^{n-1} is a synchronizing word.

Thus, the synchronizing ratio is $1 - \frac{1}{k}$.

Could the synchronizing ratio be lower?

Exhaustive search with a 32-processor computer grid

k	2	3	4	5
n	10	7	5	4

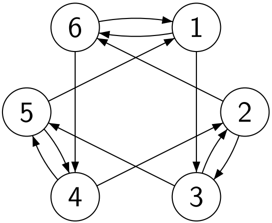
Digraph generation and evaluation :

- Generate all non-isomorphic simple graphs with `nauty`
- Orient them, add multiple edges and loops
- Leave only primitive and strongly connected
- Count the synchronizing ratio

The digraph $\mathcal{G}_{n,k}$ seems to be the worst possible with only . . .

. . . one exception!

The digraph $\mathcal{G}_{30}$.



Only 30 colorings are synchronizing out of 64 (instead of 32)

Easy to remember : the Cayley graph of S_3 with the usual generators $(0, 1)$ and $(0, 1, 2)$

Conjecture (Generalization of The Road Coloring Problem)

The synchronizing ratio of a k-out regular primitive and strongly connected digraph is at least $1 - \frac{1}{k}$, *except for* $\mathcal{G}_{30}$.

Another notable observation :

The possible numbers of synchronizing colorings are quite limited

The number of nonisomorphic digraphs with the given number of synchronizing colorings for $k = 2$ and $n = 8$

# synch. col.	128	130	...	158	160	162	...	174	176	178	180	182	184	186	188	190	192
#digraphs	72	0	...	0	24	0	...	0	1	0	0	0	5	0	0	1	813

# synch. col.	194	196	198	200	202	204	206	208	210	212	214	216	218	220	222	224	226
#digraphs	0	1	1	12	1	1	6	202	0	2	1	134	4	22	14	4,022	60

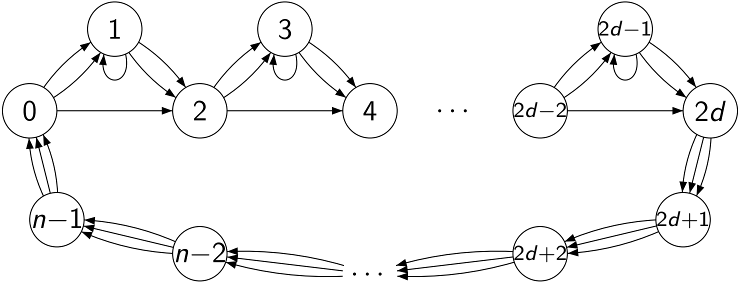
# synch. col.	228	230	232	234	236	238	240	242
#digraphs	73	170	852	179	1,226	610	21,933	699

# synch. col.	244	246	248	250	252	254	256
#digraphs	4,523	3,171	44,230	27,438	310,400	825,791	6,754,895

Conjecture

There are gaps in the distribution of the number of synchronizing colorings of k-out regular digraphs with n vertices.

The digraph $\mathcal{H}^d_{n,3}$



Theorem

The synchronizing ratio of $\mathcal{H}^d_{n,k}$ *is equal to* $1 - \frac{1}{k^d}$.

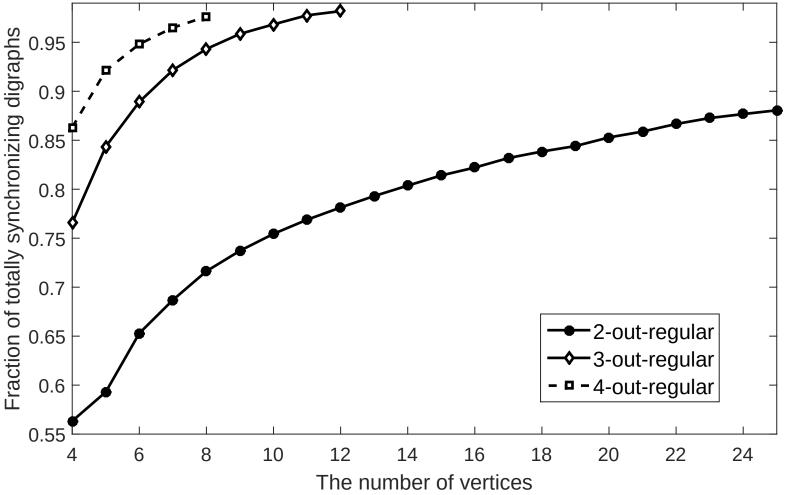
What is the average behaviour of the synchronizing ratio?

Computational experiments : it is close to 1 (there is a table in our paper)

A digraph is totally synchronizing if all of its colorings are synchronizing, i.e. the synchronizing ratio is 1.

The underlying digraphs of the Černý's automata and of the other known slowly synchronizing automata are totally synchronizing.

The fraction of totally synchronizing digraphs in the class of strongly connected and primitive digraphs



Conjecture

For every $k \geq 2$, the probability that a random k-out regular digraph is totally synchronizing goes to 1 as n goes to infinity.

Theorem (Berlinkov, 2013; Nicaud 2014)

For every $k \geq 2$, the probability that a random n-state automaton with k letters is synchronizing goes to 1 as n goes to infinity.

Random automaton $\approx$ a random digraph with a random coloring.

Is there a nice characterization of totally synchronizing digraphs?

Problem

What is the complexity of checking whether a given digraph is totally synchronizing?

The problem lies in coNP.

Problem

What is the complexity of counting the number of synchronizing colorings of a given digraph?

Conjectures :

- the length of the shortest synchronizing word is at most $(n-1)^2$
- the synchronizing ratio is at least $1 - \frac{1}{k}$ (with one exception)
- not every synchronizing ratio can be achieved
- random k-out-regular digraph is totally synchronizing

Problems :

- Complexity of checking whether a digraph is totally synchronizing
- Complexity of counting the number of synchronizing colorings