A hybrid scheme for bore design optimization of a brass instrument

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This paper presents how the shape of a brass instrument can be optimized with respect to its intonation properties. The instrument is modeled using a hybrid method between a lossy one-dimensional transmission line analogy for the slowly flaring part of the instrument, and a two-dimensional finite element model for the rapidly flaring part. The optimization employs gradient-based algorithms, and allows for a large number of design variables. Through the use of an appropriate choice of design variables, the algorithm is capable of rapidly finding horn profiles that are optimal subject to various geometric constraints, such as increasing or convex bell flares. It is found that under a convexity constraint, brass wind bells that are optimal with respect to an intonation condition can be constructed of piecewise conical sections.

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I. INTRODUCTION

The musical quality of a brass wind instrument is inseparably connected to its input impedance spectrum. Intonation and response are critically governed by the frequencies, levels, and bandwidths of the peaks in the spectrum. The input impedance is mainly determined by the variation of the duct cross-section along the instruments, and much work on how to compute the impedance for musical horns with high accuracy has been published.1–3

A problem that has started to attract interest recently, is the task to design an instrument with a given set of properties, or to suggest improvements to remedy imperfections of an already existing instrument. The purpose of this paper is to present a fast and accurate method by which it is possible to optimize the geometry of a musical horn with respect to the intonation properties controlled by the peaks in the input impedance spectrum.

Of interest is the work done by Amir et al.4 Considering wave propagation as approximately one-dimensional, they apply deconvolution methods in order to reconstruct the bore of acoustic horns from pulse reflectometry data. Similar methods for bore reconstruction have since been refined and used for historical instrument analysis.5 The work done by Bonder6 regarding the identification of the shape of the vocal tract from the formant frequencies should also be mentioned. Under idealized assumptions (one-dimensional propagation and Robin boundary conditions), wave propagation in wind instruments can be treated in the framework of Sturm–Liouville theory. The literature on inverse Sturm–Liouville problems is exhaustive, but worth mentioning is the work by Borg7 and Gladwell and Morassi.8 Although bore reconstruction methods are useful for non-intrusive measurement of the internal dimensions of existing instruments—perhaps too old and fragile for handling—they are of limited value as an aid to design or intonate instruments. A series of resonance frequencies does not uniquely determine the shape of a waveguide,9 and although it is possible to propose a whole impedance spectrum from which the instrument is then designed, it is unlikely that this would yield a geometrically acceptable instrument.

Less is reported about wind instrument design optimization in the literature. An early attempt to change the resonances of musical horns by modifying the bore was made by Amir et al.9 In a paper by Kausel,10 genetic algorithms and the Rosenbrock minimization procedure are used to optimize the intonation of brass instruments. Braden et al.11 report on different optimization criteria for trombone bore optimization. In a recent paper by Brackett et al.,12 first steps toward inclusion of structural vibrations in brass instrument optimization are taken. Le Vey13 treats the design as an optimal control problem where an internal energy functional is maximized with the horn profile as the control. Common to these papers are that they rely on a one-dimensional model for wave propagation.

The novelty of this paper is the application of a hybrid method, similar in spirit to that of Noreland,14 which uses a transmission line analogy for the narrow and slowly flaring part of the instrument, and a two-dimensional axisymmetric finite element model for the rapidly flaring part of the instrument. The hybrid method solves the problem of how to account for higher modes and how to obtain an accurate esti-
mate of the radiation impedance, while at the same time considering viscous and thermal losses. A second novelty is the use of gradient-based optimization algorithms, which renders the fast convergence necessary for computationally intensive problems with many design parameters. By the aid of the algorithm, some unorthodox, yet simple horn profiles have been found.

The rest of the paper is organized as follows. In Sec. II, the optimization problem is stated, and the objective function is formulated. The formulation is quite general with respect to the mathematical model used, anticipating the possibility to use alternative models than those proposed by the authors. Sec. III is devoted to the mathematical model of the instrument. It is also shown how the model is differentiated with respect to the design variables. In Sec. IV a case study is presented, where a French-horn-like instrument is designed from scratch. Different algorithms for finding smooth solutions are presented. The concluding Sec. V discusses various aspects of the results, and points out the advantages and limitations of the presented algorithm.

II. THE OPTIMIZATION PROBLEM

Musical instrument design is in the end an instance of multi-objective optimization in the sense that the design objectives, such as intonation, response, or cost of manufacturing, are not always commensurable. Furthermore, the design process is subject to a range of acoustical and geometrical considerations that apply either as constraints, or as factors that have to be negotiated and evaluated against each other in order to find a suitable compromise. For instance, if a section is intended to be spun over a mandrel, its diameter has to be monotonically increasing, and tuning slides are necessarily cylindrical. Furthermore, some geometries may simply be more esthetically pleasing than others. Nevertheless, desirable for the design process is a rapid optimization tool by which one can test different design objectives and geometrical constraints.

In the present work, only the musically most interesting properties of the impedance spectrum $Z_{\text{in}}(f)$, that is the peak locations and magnitudes, are addressed. The freedom left is used to find geometrically appealing shapes. Smoothness of the designs is a primary concern here. The horns are considered to be straight and of circular cross section. Bends and valves are not included, but if pre-computed impedance matrices for such objects are available, they can easily be accounted for acoustically, if not subject to the optimization. Another candidate for optimization, not addressed in this work, would be the exterior sound field.

The presented optimization problem centers around the design of an instrument with harmonically related resonances, serving as a case study. It is by no means a fact that perfect harmonic alignment is desirable, but the algorithm is general in the sense that any alignment can be prescribed. For the sake of simplicity, we restrict the study to the resonance frequencies only, although in a practical design case, the impedance magnitudes would be accounted for as well.

A. The hybrid model and optimization approach

The bore of a typical brass wind instrument can be divided into three regions, as illustrated in Fig. 1. In the narrow and slowly flaring region A, viscous and thermal effects are considerable, but very little radial motion takes place. In region C, the conditions are converse; the horn has widened enough to render viscous effects small, but the strong flare induces higher modes. In the intermediate region B, neither viscous nor two-dimensional effects are pronounced. This partitioning of effects is the basis for the hybrid method where regions A and B are analyzed using a one-dimensional viscid model, and region C is treated using a two-dimensional inviscid model. Viscosity in region C could in principle be accounted for using a more general model, such as the linearized Navier-Stokes equations, but with a substantially increased computational load due to the associated acoustic boundary layers. A simple condition for the validity of the planar wave assumption, in terms of the variation of the horn radius $R$ and the wave number $k$, is that at axial position $\ell$,

$$\delta = \frac{1}{2} \int_{0}^{\ell} kR^2(z)dz \ll 1.$$  

(1)

The number $\delta$ is an estimate of the accumulated phase error introduced by neglecting the bulging of the wavefronts as the horn expands. The validity of the assumptions can be verified a posteriori with the respective methods themselves, as detailed below.

Several runs of trial and evaluation can often be expected, and whereas the one-dimensional part can be optimized within a modest amount of time, the computational cost of a two-dimensional partial differential equation constrained optimization problem precludes interactivity. The hybridization principle of modeling is therefore carried over into the optimization scheme. The optimization is carried out in a predictor–corrector scheme consisting of two consecutive main steps. Figure 2 shows a flowchart of the process. First, a design is made using the viscid one-dimensional model throughout the length of the instrument. This is the predictor step. The region C of the obtained profile that does not, to some required level, satisfy the estimate with regard to the excitation of higher modes is then identified and re-optimized using the two-dimensional model. This second corrector step has as its goal to obtain a bell profile with the same input impedance for region C as was originally prescribed by the transmission line model. (With the software components of the algorithm it is also entirely possible to make simultaneous optimization using both parts of the hybrid model.)
A small error (roughly quantified in Sec. IV A) is inevitably introduced through neglecting viscosity in region C. When investigating the complete instrument, some of the difference between the spectrum from the one-dimensional model and the hybrid model can thus be attributed to effects other than two-dimensional. Therefore, an impedance curve computed by the one-dimensional model, but with the viscosity set to zero in region C, is used as the target impedance for the corrector step. The corrector step—based on an inviscid model—would otherwise erroneously try to remedy such viscosity-related effects by changes in the geometry. It is reasonable to assume that viscous effects for the corrected region C are to 0th order equivalent to those predicted by the lossy one-dimensional model of the uncorrected design. By computing the corrector step in the “inviscid space,” a degree of account for visco-thermal losses will thus implicitly be taken when considering the properties of the final design. A formal study of this observation is beyond the scope of this paper though, and has not been exploited in the sequel when discussing the error estimates in the model.

### B. Objective function for the predictor step

Central to any shape optimization problem is the formulation of an objective function $F$, a numerical measure of the quality of a design. In the absence of a detailed model for the lip valve and its interaction with the instrument, the frequencies of blown notes are usually identified with the resonance frequencies of the instrument with the mouthpiece rigidly sealed. If visco-thermal and radiation losses are present, the latter are in fact complex, with the imaginary part representing the damping. More specifically, the resonances are the complex frequencies $\omega$ for which the input admittance of the instrument matches the admittance of the seal, which is zero for a rigid termination. The imaginary part of these frequencies is approximated well by the commonly used locations of the impedance peaks, or by the zeros of the imaginary part along the real $\omega$-axis. As damping increases, in particular near the cut-off frequency of the horn, differences may be significant, but it should be borne in mind that a precise determination of the frequency of a blown note would have to involve properties of the lip valve, as well as the input impedance of the instrument. While acknowledging the importance of this issue, a digression is beyond the scope of this paper.

The square-sum of $\text{Im}(Z_n)$ at the desired impedance peak frequencies is one reasonable measure of the deviation in intonation. A specification of the impedance peak levels $|Z_n|$ can be included in the objective function, which also distinguishes the impedance maxima from the minima. Additional requirements, such as the value of the impedance between the peaks, can also be incorporated in $F$ in order to adjust peak bandwidths etc.

Instead of formulating the objective function from observations of $Z_n$ at fixed frequencies (in particular the desired resonance frequencies), we have chosen to employ an objective function that includes the actually observed deviations of the zeros of $\text{Im}(Z_n)$ from the desired resonances, as detailed below. This strategy improves the convergence properties of the optimization algorithm, since there is no ambiguity in the identification of the zeros with the respective resonances. Assume that the instrument shape is specified by a vector $\alpha \in \mathbb{R}^N$, the design variables, through some mathematical relation e.g., as outlined in Sec. II.D. Let $Z_n(f, \alpha)$ denote the corresponding input impedance as a function of frequency $f$. Let $f_i^r$, $i=1,\ldots,n$ be the desired (real) resonance frequencies, and denote by $f_i$ the actual $i$th zero of function $f \rightarrow \text{Im}(Z_n(f, \alpha))$. The predictor optimization problem can then be formulated: find

$$
\min_{\alpha} F_p(\alpha), \quad F_p(\alpha) = \frac{1}{2} \sum_{i=1}^{n} (f_i - f_i^r)^2
$$

where

$$
\text{Im}(Z_n(f_i, \alpha)) = 0, \quad i = 1, \ldots, n.
$$

Note that, if desired, terms of the type $(|Z_n(f_i, \alpha)| - Z_i^r)^2$ easily can be added to the objective function in order to optimize also with respect to the peak impedance values. Problem (2) is a non-convex least-squares problem that can be solved efficiently using a gradient based method, provided that the starting solution is in some sense close to an optimal design. This is the case in many practical situations, for instance when optimizing the intonation. The algorithm chosen for the predictor step optimization is the Levenberg–Marquardt method as implemented in the function lsqnonlin in the optimization suite of MATLAB.

The computation of the objective function $F_p$ and its derivatives required by the optimization algorithm is carried out in the following way.

1. Find the zeros $f_i$, $i=1,\ldots,n$ of $f \rightarrow \text{Im}(Z_n(f, \alpha))$.
2. Form vector $r_i = f_i - f_i^r$.
3. Calculate $F_p(\alpha) = \frac{1}{2} \sum r_i^2$.
4. Form $\frac{dr_i}{d\alpha_j} = -\text{Im}(\partial Z_n(f_i, \alpha)/\partial \alpha_j)/\text{Im}(\partial Z_n(f_i, \alpha)/\partial f)$.

The information supplied to the least-squares solver is mainly the function $r(\alpha)$ and the Jacobian matrix $\nabla r(\alpha)^T$. Step 4, defining the gradient components, follows from differentiation of equation $\text{Im}(Z_n(f_i(\alpha), \alpha)) = 0$ with respect to $\alpha_j$, which gives
The implementation of step 1 necessitates a global root finding algorithm that automatically finds the \( n \) first zeros of function \( f \rightarrow \text{Im}(Z_m(f, \alpha)) \) corresponding to impedance maxima. Note that it is necessary to differentiate \( Z_m \) with respect to both frequency \( f \) and the design variables \( \alpha \). The process of finding the roots is computationally more involved than merely evaluating \( Z_m \) at the desired peak frequencies, but the extra cost is justified for this more general design approach.

C. Objective function for the corrector step

The two-dimensional corrector step is formulated differently in that the objective function is more globally dependent on the impedance curve. This makes it in effect an inverse problem, although design constraints (increasing, convex etc.) of much the same kind as in the one-dimensional step are imposed.

We choose to base the objective function for the bell optimization on the complex reflection factor instead of the bell input impedance. Since these quantities are related through the conformal mappings

\[
R = \frac{Z_m - \rho c/S}{Z_m + \rho c/S}, \quad Z_m = \frac{1 + R}{1 - R} \rho c/S,
\]

where \( \rho \), \( c \), and \( S \) are the density of air, speed of sound, and input cross-sectional area, it is merely a matter of convenience whether one or the other quantity is used in the optimization. The use of the reflection factor avoids the quasi-singularity of the input impedance at resonance tops, which is why we expected it to be numerically advantageous. Considering a frequency band \([0, \omega_{\text{max}}]\), a suitable objective function is

\[
F_c(\alpha) = \frac{1}{2} \int_0^{\omega_{\text{max}}} w(\omega)|R(\alpha, \omega) - R_{1D}(\omega)|^2 d\omega,
\]

where \( R(\alpha, \omega) \) is the reflection function for a bell profile described by design variables \( \alpha \), and \( R_{1D}(\omega) \) is the target function as given by the one-dimensional model. The function \( w \) is used to put extra weight on critical features, such as sharp peaks.

D. Shape representations in the optimization

Diameters \( y_1, \ldots, y_{N+1} \) and segment lengths \( L_1, \ldots, L_N \) in Fig. 3 completely specify the geometry of the instrument. The actual choice of segment lengths and \( N \) is made with regard to what is a necessary geometrical resolution with respect to modeling accuracy and manufacturing factors.

In the case study presented in section IV, the one-dimensional (1-D) optimization step was carried out using as design variables \( \alpha = (L_1, \ldots, L_N) \), with \( \alpha \geq 0 \), keeping a pre-defined set of increasing diameters \( y_1, \ldots, y_{N+1} \) fixed throughout the optimization. The resulting design is then guaranteed to be monotonically increasing.

For the two-dimensional corrector step, three classes of admissible bell shapes were investigated: (i) increasing shapes, (ii) unconstrained curvature-parameterized shapes, and (iii) convexity-constrained curvature-parameterized shapes.

Let now the horn in Fig. 3 depict the bell region \( C \) subject to the corrector design step. Alternative (i), the increasing shapes, are specified by using \( \alpha = (y_2 - y_1, y_3 - y_2, \ldots, y_{N+1} - y_N) \), with \( \alpha \geq 0 \), as design variables, keeping a predefined set of segment lengths \( L_1, \ldots, L_N \) fixed throughout the optimization. The bell diameter \( y_{N+1} \) as well as the inlet diameter \( y_1 \) are kept constant. Therefore, the constraint \( \sum_{i=1}^{N} \alpha_i = y_{N+1} - y_1 \) needs to be enforced by the optimization algorithm in this case.

The curvature-parameterized shapes of alternatives (ii) and (iii) are based on an indirect specification of the function \( \beta \) that represents the displacement of the bell contour with respect to a straight conical shape generated by the line segment \( \Gamma_{d} \), as illustrated in Fig. 4. The function \( \beta \) is related to a function \( \alpha \), which after discretization becomes the actual design variable, through the 1-D boundary-value problem \(-\beta = \alpha \) on \( \Gamma_{d} \) with vanishing boundary condition for \( \beta \) at the end points of \( \Gamma_{d} \). Discretization with finite elements (or finite differences) results in a system of equations \( K_{\beta} \beta = \alpha \), where the elements of vector \( \beta \) yields the normal displacement of the bell at the nodes of a division of \( \Gamma_{d} \), and where matrix \( K_{\beta} \) is a 1-D discrete Laplacian operator. The design variables \( \alpha \) represent here roughly the local curvature of the bell shape, which motivates the name of the parameterization.
The curvature parameterization has a number of useful properties. It promotes smooth design updates and prevents the optimization algorithm from being caught in local optima associated with non-smooth boundary shapes. The convex flare constraint of alternative (iii) is easily implemented by requiring $\alpha \geq 0$. Adding the so-called Tikhonov regularization term $\varepsilon \| \alpha \|^2/2$ to the objective function yields a convenient way of enforcing even smoother shapes. The properties of the curvature parameterization have been thoroughly investigated in the case of loudspeaker horn optimization in the article by Bångtsson et al., which provides additional details about the method.

III. MATHEMATICAL MODELS

This section briefly outlines the 1-D transmission line model, the two-dimensional (2-D) bell model for the reconstruction step, and how the derivatives needed for the optimization algorithms are computed.

A. 1-D: Transmission line model

Technical details on parts of the one-dimensional optimization scheme have previously been presented by Noreland, but the process is briefly outlined here. Wave propagation in region A and B can be described by Webster’s horn equation, which is solved numerically by the so-called transmission line (TL) analogy, whereby the waveguide is discretized as a succession of waveguide elements (Fig. 3), each with a known analytical expression for its transfer matrix. The transfer matrix of the whole waveguide is then found as the product of the individual element matrices. Visco-thermal losses are accounted for using the appropriate data structures, the true wavefronts are neither planar nor spherical. At any rate, the Webster equation is an approximation, and the results should be in close agreement. Other formulations of one-parameter waves are conceivable, but a requisite is that the shape of the end interface that matches the entry of region C is frequency independent.

The input impedance $Z_{in}(f)$ relates the acoustic volume flow velocity, $U_{in}(f)$, to the acoustic pressure, $p_{in}(f)$, at the mouthpiece of the instrument through the relation $p_{in}(f) = Z_{in}(f)U_{in}(f)$. Denoting by $\mathbf{H}$ the $2 \times 2$ transfer matrix of the instrument that relate input and output pressure and volume flow velocity through $[p_{in},U_{in}] = \mathbf{H}[p_{out},U_{out}]$, the input impedance seen from the mouthpiece is given by

$$Z_{in} = \frac{H_{12} + H_{11}Z_{L}}{H_{22} + H_{21}Z_{L}}.$$  

where $Z_{L} = p_{out}/U_{out}$ is the radiation impedance seen from the end of the horn. In the TL model, the instrument is regarded as a series of conical waveguide elements according to Fig. 3. The elements are numbered $1, 2, \ldots, N$, and the element lengths are denoted $L_1, L_2, \ldots, L_N$. The end diameters at the junctions are denoted $y_1, y_2, \ldots, y_{N+1}$. The transmission matrix of the whole structure is given by the product of the individual transfer matrices $\mathbf{H}_j$, $j = 1, \ldots, N$ of each segment:

$$\mathbf{H}(f) = \prod_{j=1}^N \mathbf{H}_j(f; y_j, y_{j+1}, L_j).$$  

For the elements of $\mathbf{H}_j$ we refer e.g., to a paper by Mapes-Riordan.

B. 1-D: Derivatives of $Z_{in}$

An efficient way of computing the derivatives of the input impedance $Z_{in}$ with respect to the design variables is necessary for the chosen optimization algorithm. In some cases, when the function is complicated, the gradient must be approximated using finite differencing for each of the design variables. This may occasionally be detrimental to the convergence of the optimization algorithm due to the truncation error in the gradient, but the preponderant drawback is that the work for computing the gradient grows as the square of the number of discretization segments, soon making finite differences infeasible. Instead, by symbolic manipulation and using the appropriate data structures, the gradient of $Z_{in}$ can be computed at a cost proportional to the number of segments in the model. The derivative of Eq. (4) with respect to $L_j$ requires the differentiation of $\mathbf{H}$. For this, define

$$\Pi_{ij} = \mathbf{H}_i \mathbf{H}_{i+1} \cdots \mathbf{H}_j, \quad j > i.$$

Only segment $j$ is affected by a change of $L_j$. Hence, for $2 \leq j \leq N-1,$

$$\frac{\partial \mathbf{H}}{\partial L_j} = \Pi_{i,j-1} \frac{\partial \mathbf{H}}{\partial L_j} \Pi_{j+1,N}.$$  

with obvious extensions to the cases $j = 1, N$. Differentiation with respect to $y_k$ is implemented accordingly, noting that two adjacent segments depend on $y_k$. Differentiation with respect to $f$ can also be done along these lines, but it is more economical to use a finite difference approximation. Computing $\Pi_{1,1}, \Pi_{1,2}, \ldots, \Pi_{N-1,N}, \Pi_{N,N}, \Pi_{N-1,N}, \ldots, \Pi_{2,N}$ requires a total of $2N-3$ matrix multiplications. Utilizing the appropriate data structures for the cumulative products involved in Eq. (6), the total work of computing the gradient is around eight times that of evaluating $Z_{in}$ by propagating an expression of the type (4) through the waveguide.

C. 2-D: Bell model

The input impedance of bell region C is computed by simulating the wave propagation under arbitrary excitation at
the inlet opening. We model the wave propagation by the inviscid Helmholtz equation in cylindrical symmetry. Figure 4 depicts the computational region.

The region exterior to the bell is truncated to a bounded domain $\Omega$ to set up for the numerical solution, where an artificial boundary condition of Engquist–Majda type at distance $R_h$ approximates the far-field Sommerfeld condition. Moreover, at the inlet boundary $\Gamma_{in}$, where higher modes are negligible by construction, a right-going planar wave of amplitude $A$ is enforced, whereas the left-going wave is assumed perfectly absorbed. The bell walls $\Gamma_n$ are made of sound hard material. Assuming an $\exp(i\omega t)$ dependence, these assumptions lead to the boundary value problem

$$\begin{align*}
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial p}{\partial r} \right) + \frac{\partial^2 p}{\partial z^2} + k^2 p &= 0 \quad \text{in } \Omega, \\
(k + \frac{1}{R_\Omega}) p + \frac{\partial p}{\partial n} &= 0 \quad \text{on } \Gamma_{out}, \\
i k p + \frac{\partial p}{\partial n} &= 2i k A \quad \text{on } \Gamma_{in},
\end{align*}$$

where $k = \omega / c$.

Boundary value problem (7) is solved numerically using a finite-element method employing continuous, piecewise-linear functions on an unstructured triangular mesh covering $\Omega$. The discretization results in the following linear system of equations for the vector $p$ of nodal pressures:

$$(K - k^2 M + N_{\text{bnd}})p = f_{in},$$

where stiffness and mass matrices $K$ and $M$ depend on the design boundary shape $\Gamma_d$. Matrix $N_{\text{bnd}}$ originates from the boundary conditions on $\Gamma_{in}$ and $\Gamma_{out}$, and the right hand side $f_{in}$ is the term responsible for the handle of an incoming wave at $\Gamma_{in}$. For details on the discretization, we refer to the article by Udawalpola and Berggren, where the same system is used for modeling loudspeaker horns.

D. 2-D: Objective function, design alterations, and gradient computations

Assume now that Eq. (7) is solved for a particular value of the design variable $\alpha$ and at angular frequency $\omega$. With an incoming wave of amplitude $A=1$ generated at $\Gamma_{in}$, assumed to be placed at axial coordinate $z=0$, the pressure field in the vicinity of $\Gamma_{in}$ can be written

$$p(r, z; \alpha, \omega) = \exp(-ikz) + R(\alpha; \omega) \exp(ikz).$$

The amplitude of the reflected wave at $\Gamma_{in}$ will then be

$$R(\alpha; \omega) = p(r, 0; \alpha, \omega) - 1,$$

and a discrete approximation of objective function (3) is

$$F(\alpha) = \frac{1}{2} \sum_k w_k |p(r, 0; \alpha, \omega) - 1 - R_{1D}(\omega)|^2.$$  

When the bell flare is modified using one of the three strategies (increasing, unconstrained, or convex shapes) outlined in Sec. II D, the computational mesh inside domain $\Omega$ also has to be altered in order to retain the quality of mesh. A complete re-meshing at each design cycle is not used, since it would likely lead to “numerical noise,” sudden small random changes in the objective function due to topology changes of the mesh, affecting the robustness of the optimization algorithm. Instead, the displacements of the mesh vertices at the design boundary are transferred to displacements of the mesh vertices in the interior of $\Omega$ using an elliptic smoother. This strategy results in a continuous deformation of the original mesh that smoothly follows the flare changes.

Each cycle of modification of design variables $\alpha$ requires thus the following five steps in the 2-D analysis:

1. Using one of the three strategies outlined in Sec. II D, the new vector of design variables $\alpha$ is translated into modifications of the locations of the mesh vertices on the design boundary.
2. The locations of the mesh vertices internal to $\Omega$ are modified using the elliptic smoother.
3. The mass and stiffness matrices of Eq. (8) are recalculated.
4. Equation (8) is solved.
5. Objective function (9) is evaluated.

The success of the optimization is also here crucially dependent on accurate and efficient computation of the gradient of objective function (9). Through the so-called adjoint technique, the gradient of the compound map associated with the five steps above can be computed at a cost that is essentially independent of the dimension of design variable $\alpha$. Giles and Pierce provide a general introduction to the adjoint method for design optimization in a fluid mechanics context (but the general principles are the same in this case).

The details of the implementation of the adjoint method for the current case are quite involved and we refer to earlier publications for the details.

The large-scale version of MATLAB’s lsqnonlin is the optimization algorithm used for the 2-D corrector step when employing the curvature parameterization. This nonlinear least-squares algorithm is capable of handling the bound constraints $\alpha \geq 0$ that yields convexity. However, the equality constraint $\sum_{i=1}^{N} \alpha_i = 1$ is needed and the increasing shapes, cannot be handled by lsqnonlin. The Method of Moving Asymptotes by Svanberg, which can handle quite general types of constraints, was therefore used for the increasing shapes.

IV. DESIGN OF A HORN-LIKE INSTRUMENT

In this case study, the task is to design from scratch a French-horn-like instrument with the first 16 resonances at integer multiples of 50 Hz. The first resonance is placed at 35 Hz, however, which distinguishes the result from a more tuba-like instrument, where also the first resonance is part of the harmonic series. For the sake of simplicity, no mouth-piece is included. The geometry description, in terms of the parameters $L_1, \ldots, L_N$, and $y_1, \ldots, y_{N+1}$ in Fig. 3, uses $N=61$ segments.

In the first step of the hybrid algorithm, monotonicity is enforced by starting with a monotonously increasing horn,
and letting $\alpha=(L_1, L_2, \ldots, L_N)^T$. Figure 5 shows the initial geometry made up of a cylindrical piece of tubing attached to a generic horn. The resonances for this start design are off by up to 130 Hz, and the spacing between the resonances is between 18 Hz too small and 19 Hz too large; a mere scaling of the cylinder and/or the bell in the axial direction is insufficient to achieve any reasonable degree of intonation.

The optimization terminated for a design whose resonances lie within 0.07 Hz of the target frequencies, for all of the 16 notes. The optimized horn is also shown in Fig. 5. The cylinder has been prolonged, but we also note that the flare profile has changed. Apart from this, the instrument has a rather orthodox shape.

The variation of the integral in Eq. (1) along the horn axis is used as an indicator for the position of the interface to region C. We have chosen to use the 2-D model from the point where $\delta=10^{-3}$ at 850 Hz. Translated into a length error, this accumulated phase error between the mouthpiece end and the interface point is a negligible 0.07 mm. Accordingly, the switch to the two-dimensional model is made for the part where radius $r=17$ mm. A control of the rest of the horn using the 2-D finite element (FE) model for reference duly shows the strengths and the limitation of one-dimensional modeling. Shown in Fig. 6 are the impedance curves of the uncorrected bell design (region C) according to the TL model with and without viscosity, and according to the FE model. The agreement is good until 440 Hz, when the TL curves start to deviate from the FE curve. The results indicate also that viscosity is of relatively limited importance in region C; only at the narrow-band ($Q=33$) first peak is the difference between viscous and non-viscous models apparent with a 62% too high peak for the latter. In terms of the complete

![Fig. 5. Initial (⋯) and optimized (−) horn profile computed with the transmission line model. (Part of the leading cylindrical section omitted for clarity.)](image)

![Fig. 6. Input impedance for the bell (region C) according to the inviscid TL model (−⋯), viscous TL model (⋯⋯) and the FE model (−).](image)

![Fig. 7. $Z_{in}$ spectrum (a) and shape of region C (b) for unconstrained optimization with $\epsilon=10^{-3}$ (⋯⋯) and $\epsilon=10^{-2}$ (⋯).](image)

![Table I. Intonation of resonances in musical cents for the different designs.](image)
amount of regularization. The contour is markedly smoother in the latter case, though, as can be seen in Fig. 7(b).

Figure 8(b) shows the result when constraining the shapes to be increasing. The profile is characterized by four rapid expansions between which the horn is cylindrical. Figure 8(a) shows a surprisingly good correspondence with the target impedance.

Although modern manufacturing techniques make it possible to make also contracting/expanding horns, it is likely that a convex horn easier would find immediate acceptance among musicians. Figures 9(a) and 9(b) show the results when using convexity-constrained curvature-parameterized shapes. (A small degree of regularization was also included for technical reasons.) A striking piecewise conical shape was achieved. In order to gain insight into the process governing the formation of a discontinuous contour derivative, the design process was repeated with the maximum frequency in objective function (9) limited to 400 Hz. Since the 1-D design from step 1 was not completely convex, some design changes were bound to take place. Figure 10(b) shows that the resulting shape is more than a mere “convexification” of the start design, but no corners like in the previous optimization run appear. Thus, the corners appear when attempting to match also the high end of the target spectrum in objective function (9).

For comparison, the best possible Bessel horn with the same end diameters and length as the other bells was computed. With these constraints, there is only one free parameter left. The design is not as inferior as the table indicates. The first 14 resonances have a rms difference of 6.2 cents, and none of those is off by more than 11 cents. It is largely the high frequency breakdown that is responsible for the bad figures.

Input impedance spectra for the complete instrument are presented in Fig. 11. Shown are the target spectrum as proposed by the predictor step, the spectrum of the uncorrected design as computed by the hybrid method, and the spectrum after applying the corrector step with the convexity constraint imposed. The curves agree well up to approximately 450 Hz, when the impedance of the uncorrected design starts to diverge from the target. The corrected bell does to some extent remedy the defects of the uncorrected design, and although the resonance frequencies have been improved, deviations are rather substantial in the high frequency band. A comparison (not presented in the figure) with results for any of the other sets of constraints shows a much closer agreement between target and corrected curves.

A. Error estimate

There are three kinds of errors associated to the used hybrid scheme: (i) numerical errors, (ii) errors due to the assumption that viscosity is effectively zero in part of the horn and (iii) the one-dimensional approximation of propagation. The numerical errors are governed by discretization errors, and errors due to the finite computational domain. A reference computation using higher order elements (or a refined mesh) and a varying $R_{le}$ verified that these errors are
would probably be possible to reduce the size of C.

small. The exact influence of the neglected viscosity in the bell cannot be computed with the presented model, but an estimate using only the TL model shows that the neglected viscosity in region C perturbs the 16 resonances by 1 cent rms (maximum 2 cents). The peak magnitudes is perturbed by 3% rms (maximum 9%). The error due to the one-dimensional assumption in region A-B is very small according to Eq. (1), but due to the simple nature of this error estimate, we chose to err on the side of caution when selecting the size of region C. With a more precise estimate, it would probably be possible to reduce the size of C.

The two-dimensional optimization part has been validated by numerical experiments with the impedance curve of a known, smooth bell profile as the target impedance. Starting the optimization from an initial design represented by a perturbation from the known shape, the original shape was quickly recovered, which proves the scheme to be at least locally convergent.

V. CONCLUSIONS

While the one-dimensional TL-model is quite accurate in determining low and mid-range resonance frequencies, a more accurate method is called for at higher frequencies. The FE model is attractive in that it offers a general account of higher modes. It allows in principle for inclusion of additional objects such as a mute, or a variable speed of sound. Its preponderant drawback is the lack of a model for viscothermal losses. As indicated in sec. II A, this deficiency is probably largely compensated for through the choice of target impedance in the corrector step. A quantitative examination of these effects is however left for a future study.

The proposed optimization scheme is both comparatively fast and robust against development of unacceptable shapes. Although the optimization of the rapidly flaring part of the bell is performed in two dimensions, the computational effort necessary is not heavier than it can be done in the course of a couple of hours on a personal computer with good performance. The one-dimensional part, in turn, runs quickly enough for interactive work (a few seconds for 100 design variables).

There are several ways to tackle the problem of how to find geometrically acceptable shapes. The inclusion of a Tikhonov regularization term has been treated. Another possibility, which is commonly used in other applications, is to optimize with respect to a number of variables in a CAD (computer-aided design) description of the geometry, for instance the control points of a Bézier spline. However, a shape such as the one in Fig. 10(b) would be difficult to discover with such a parameterization. This geometry, where the bell is made up of a number of truncated cones, could arguably be described by a parameterization with very few variables. While this is clear a posteriori, there is no known way to determine the number of cones in advance, or even to forecast that this would be an advantageous kind of shape. By the use of a large number of degrees of freedom, it was possible to discover this surprising fact.

In more applied terms, the piecewise conical shape might look intimidating, but from a manufacturing point of view it would be very easy to produce such a bell. It is not entirely clear why the optimization is prone to develop such piecewise linear profiles when the convexity constraint is imposed. When a discrete set of frequencies is considered—as is the case for the whole instrument—it is at first glance not surprising if localized changes such as step discontinuities develop. However, the secondary 2-D design of the bell looks at a densely sampled impedance impedance curve, so there is at least no explicit coupling to the set of resonance frequencies. Apart from the numerical experiments presented in this paper, numerous variations have been carried out. The tendency to develop piecewise conicity is
corners in the bell, as would relaxing the conditions regarding the search space for region A and B, for instance including a be fixed. While these are reasonable engineering constraints, the optimization scheme assumes the end diameter and bell length to sate for the high frequency end-effects. be seen as a desperate measure of the optimizer to compen-

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terms of localized changes in the optimization parameter.

In terms of intonation performance of the instrument, the designs rank in the falling order increasing, unconstrained curvature-parameterized, convexity-constrained curvature-parameterized, and Bessel, respectively. One would perhaps expect the unconstrained curvature-parameterized shape to outperform the increasing shape, since the latter is in a subspace of the former, but the difference is small and a different choice of would might change the outcome. Table I should also be interpreted with some caution, since it does not say anything about the behavior of the impedance between the peaks. It should also be noted that in general, only local optima can be found by the used optimization methods; there are many quite different shapes that provide good matching, and convergence is more likely toward solutions in the vicinity of the initial design than far away. The interpretation of vicinities must however be made in terms of the optimization variable, and not directly in the visual appearance of the profile. Nevertheless, for any set of constraints, regularization improves smoothness at the expense of acoustical quality. By and large, the ranking is not surprising, since for shapes of positive opening angle at the throat, they are subsets of one another, and adding constraints compromises the acoustical properties.

It is long known that infinitely flaring Bessel horns show perfectly harmonic resonances under the assumption of inviscid plane wave propagation. It should thus come as no surprise that it is easy to find harmonic horns with a smooth profile considering the low and mid frequency range, since then at least the planar wave assumption is quite accurate. As the frequency increases, a one-dimensional model becomes less and less valid, and in order to counteract the perturbations, the optimization has to apply deformations that deviate from a smoothly flaring horn. Allowing also for contracting horns, this is readily done, but the convexity constraint is really quite restrictive. The formation of corners, or a stepped conical bore as in the increasing shape example, can be seen as a desperate measure of the optimizer to compensate for the high frequency end-effects.

The current formulation of the two-dimensional optimization scheme assumes the end diameter and bell length to be fixed. While these are reasonable engineering constraints, a more general solution would be to let also these dimensions vary. It should also be pointed out that an enlarged search space for region A and B, for instance including a number of local constrictions, might eliminate the need for corners in the bell, as would relaxing the conditions regarding peak magnitudes that are implicitly considered in the corrector step.