Parallel solution of banded and block bidiagonal linear systems

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Outline of Topics

Banded systems
- Motivation
- Diagonally dominant systems
- General banded systems

Cyclic reduction of BBD systems
- Motivation
- Serial code
- Parallel code
Motivation

Why bother?

- Naturally occurring.
Motivation

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- Naturally occurring.
- Reordering of sparse matrices.
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- Naturally occurring.
- Reordering of sparse matrices.
- Banded preconditioners.
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Parallel solution of banded and block bidiagonal linear systems
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Parallel solution of banded and block bidiagonal linear systems
The half bandwidth $k$ of the matrix satisfies

$$|i - j| > k \Rightarrow a_{ij} = 0.$$
Notation

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  \[ |i - j| > k \implies a_{ij} = 0. \]

- A matrix $A$ is (strictly) diagonally dominant by rows, if
  \[ \forall i : \sum_{j \neq i} |a_{ij}| < |a_{ii}|. \]
Notation

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  \]

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  \forall \; i : \sum_{j \neq i} |a_{ij}| < |a_{ii}|.
  \]

- The dominance factor \( \epsilon \) is defined by
  \[
  \epsilon = \max_i \left\{ \frac{\sum_{j \neq i} |a_{ij}|}{|a_{ii}|} \right\} \in [0, 1).
  \]
Theorem (Mikkelsen, 2010)

Let $A$ be a matrix which is strictly diagonally dominant by rows, let $S$ be the Schur complement for the ScaLAPACK matrix, and let $D$ be the main block diagonal of $S$. Let

$$\mu = qk$$

denote dimension of the individual partitions. Then

$$\|S - D\|_\infty \leq 2\epsilon^{1+q}\|A\|_\infty$$

where $\epsilon$ is the dominance factor.
The origins of the off diagonal blocks
The central estimate

\[ \epsilon^q = n = qk \]

**Figure:** The $\| \cdot \|_\infty$ norm of the blue $k$ by $k$ blocks decay exponentially.
Overview

- Parallelism is achieved by reordering.
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- The Schur complement is factored using cyclic reduction.
Overview

- Parallelism is achieved by reordering.
- The Schur complement is block bidiagonal.
- The Schur complement is factored using cyclic reduction.
- The routines are designed for narrow banded matrices.
Why bother?

- Central to ScaLAPACKs banded solvers.
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- Block bidiagonal (BBD) systems are interesting on their own.
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- Central to ScaLAPACKs banded solvers.
- Block bidiagonal (BBD) systems are interesting on their own.
- The LU factorization may fail, where the QR factorization succeeds.
Figure: Hand calculation versus computer implementation
Figure: Apply a block column permutation to the leaves
Figure: The recursion begins
Figure: We reach the leftmost leaf
Figure: QR factor the blue columns and ...
Figure: ... apply the orthonormal transformation to the red columns
Figure: Branch completed
Figure: QR factor the blue columns ...
Figure: ... apply the orthonormal transformation to the red columns
Figure: Branch completed
Figure: Identify data for transport
Figure: Move data upward
Figure: Transfer complete
Figure: QR factor the blue columns and ...
Figure: ... apply the orthogonal transformation to the red columns
Figure: Branch completed
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Figure: QR factor the blue columns ...
Figure: ... apply the orthogonal transformation to the red columns
Figure: Branch completed
Figure: Identify data for transport
Figure: Move data upward
Figure: Transfer complete
Figure: QR factor the blue columns ...
Figure: ... and we are done!
Figure: Tree completed
Figure: Permute rows 4 and 5
Figure: Permute columns 4 and 5
Figure: Permute rows 2 and 3
Figure: Permute columns 2 and 3
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- **Banded systems**
  - Cyclic reduction of BBD systems

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#### Example Diagram

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Current implementation

- Parallel solution of BBD systems using the QR factorization
Overview

Current implementation

- Parallel solution of BBD systems using the QR factorization
- Fortran 90
Overview

Current implementation

- Parallel solution of BBD systems using the QR factorization
- Fortran 90
- Zero interprocessor communication outside of ScaLAPACK.
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Thank you for your attention!