COMPARING ONE-SHOT AND MULTI-SHOT METHODS FOR SOLVING PERIODIC RICCATI DIFFERENTIAL EQUATIONS [1]

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Abstract: One-shot methods and recently proposed multi-shot methods for computing stabilizing solutions of continuous-time periodic Riccati differential equations are examined and evaluated on two test problems: (i) a stabilization problem for an artificially constructed time-varying linear system for which the exact solution is known; (ii) a nonlinear stabilization problem for a devil stick juggling model along a periodic trajectory. The numerical comparisons are performed using both general purpose and symplectic integration methods for solving the associated Hamiltonian differential systems. In the multi-shot method a stable subspace is determined using recent algorithms for computing a reordered periodic real Schur form. The results show the increased accuracy achievable by combining multi-shot methods with structure preserving (symplectic) integration techniques.

Keywords: periodic systems, reordered periodic Schur form, Riccati differential equations, stabilizing controllers, linear quadratic regulators

1. INTRODUCTION

In this contribution, we consider the computation of stabilizing controllers for linear periodic time-varying systems

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \qquad (1)$$

where $A(t) \in \mathbb{R}^{n\times n}$ and $B(t) \in \mathbb{R}^{n\times m}$ are T-periodic matrices, i.e., $A(t) = A(t+T)$ and $B(t) = B(t+T)$ for all t. The optimal periodic controller is given via solving the *linear quadratic regulator* (LQR) problem, i.e., by minimizing the quadratic cost function for (1):

$$\min_{u(t)} \int_0^\infty \left[x(t)^T Q(t)x(t) + u(t)^T \Gamma(t)u(t)\right] dt, \qquad (2)$$

where $Q(t) \in \mathbb{R}^{n\times n}$ and $\Gamma(t) \in \mathbb{R}^{m\times m}$ are T-periodic matrices, and $Q(t) = Q(t)^T \geq 0$ (symmetric positive semidefinite) and $\Gamma(t) = \Gamma(t)^T > 0$ (symmetric positive definite) for all t. Provided the pair $(A(t), B(t))$ is stabilizable and $(A(t), Q(t)^{1/2})$ is detectable, where $\left(Q(t)^{1/2}\right)^T Q(t)^{1/2} = Q(t)$, the optimal periodic feedback $u^*(t)$ that stabilizes (1) and minimizes (2) is

$$u^*(t) = -K(t)x(t), \qquad (3)$$

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where $K(t) = \Gamma(t)^{-1}B(t)^T X(t)$.

The periodic matrix $X(t)$ in (3) is the unique symmetric positive semidefinite T-periodic stabilizing solution of the continuous-time *periodic Riccati differential equation* (PRDE) (Bittanti, 1991; Yakubovich, 1986):

$$\begin{aligned} -\dot{X}(t) &= A(t)^T X(t) + X(t)A(t) \\ &\quad - X(t)B(t)\Gamma(t)^{-1}B(t)^T X(t) + Q(t). \end{aligned} \tag{4}$$

In the following, two methods to solve the PRDE (4) are examined: the one-shot periodic generator method (e.g., see (Hench *et al.*, 1994)) and a multi-shot method proposed in (Varga, 2005).

2. ONE-SHOT METHOD

Let $H(t) \in \mathbb{R}^{2n\times 2n}$ be a time-varying Hamiltonian matrix defined as

$$H(t) = \begin{bmatrix} A(t) & -B(t)\Gamma(t)^{-1}B(t)^T \\ -Q(t) & -A(t)^T \end{bmatrix},$$

i.e., $H(t)$ satisfies $JH(t)^T J^T = -H(t)$ for all t, where $J = \left[\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right]$. From the initial value problem

$$\frac{\partial}{\partial t}\Phi(t,t_0) = H(t)\Phi(t,t_0), \quad \Phi(t_0,t_0) = I_{2n}, \tag{5}$$

the *transition matrix* $\Phi(t,t_0)$ associated with $H(t)$ is computed. The system (5) is a linear Hamiltonian system where the transition matrix $\Phi(t,t_0)$ for all $t > t_0$ has eigenvalues symmetric with respect to the unit circle and is symplectic, i.e., $J^T\Phi(t,t_0)^T J = \Phi(t,t_0)^{-1} = \Phi(t_0,t)$ for all t (Leimkuhler and Reich, 2004). For a T-periodic system, the transition matrix evaluated over one period is known as the *monodromy matrix* $\Psi(t_0) = \Phi(t_0+T,t_0)$.

The stabilizing solution for a PRDE (4) is obtained by the following approach (Bittanti, 1991; Hench *et al.*, 1994; Yakubovich, 1986):

(1) Compute the monodromy matrix $\Psi(t_0) = \Phi(t_0+T,t_0)$ by solving the initial value problem (5) over one period.

(2) Compute the ordered real Schur form of $\Psi(t_0)$:

$$\begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}^T \Psi(t_0) \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix},$$

where $S_{11} \in \mathbb{R}^{n\times n}$ is upper quasi-triangular with n eigenvalues inside the unit circle, and $S_{22} \in \mathbb{R}^{n\times n}$ is upper quasi-triangular with n eigenvalues outside the unit circle[2]. Then the stable subspace of $\Psi(t_0)$ is spanned by the columns of the $2n\times n$ matrix $\begin{bmatrix} U_{11} \\ U_{21} \end{bmatrix}$.

(3) Solve the matrix differential equation

$$\dot{Y}(t) = H(t)Y(t), \quad Y(t_0) = \begin{bmatrix} U_{11} \\ U_{21} \end{bmatrix}, \tag{6}$$

by integrating from $t = t_0$ to $t = t_0 + T$.

(4) Partition the solution of (6) into $n\times n$ blocks:

$$Y(t) = \begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix}.$$

Then the solution of the PRDE is computed:

$$X(t) = Y_2(t)Y_1(t)^{-1}, \quad t = t_0,\ldots,t_0+T.$$

In step 1, it is important to use a symplectic integrator, which is confirmed by the numerical experiments in Section 4. The one-shot periodic generator method solves an ODE with unstable dynamics in both steps 1 and 3, and therefore this method is unreliable for systems with large periods (Varga, 2005).

3. MULTI-SHOT METHOD

As an alternative to the one-shot method, we consider the multi-shot method proposed in (Varga, 2005). The main idea is to turn the continuous-time problem into an equivalent discrete-time problem. Instead of integrating (5) over one whole period, the monodromy matrix $\Psi(t_0)$ is computed using the following product form of the transition matrix (for simplicity, let $t_0 = 0$): $\Psi(0) = \Phi(T,0) = \Phi(T,T-\Delta)\cdots\Phi(2\Delta,\Delta)\Phi(\Delta,0)$, where $\Delta = T/N$ for a suitable integer N. In the following, let Φ_k denote the transition matrices, i.e., $\Phi_k = \Phi(k\Delta,(k-1)\Delta)$ for $k = 1,\ldots,N$.

To compute the stable subspace of $\Psi(0)$ the *periodic real Schur form* (PRSF) is used (Bojanczyk *et al.*, 1992; Hench and Laub, 1994): For an arbitrary real matrix sequence $A_1, A_2, \ldots, A_N$ there exists an orthogonal matrix sequence $Z_k \in \mathbb{R}^{n\times n}$:

$$Z_{k+1}^T A_k Z_k = S_k, \quad k = 1,\ldots,N,$$

with $Z_{N+1} = Z_1$ and where one of the S_k matrices, say S_r, is upper quasi-triangular and the remaining $N-1$ are upper triangular. The quasi-triangular S_r has 1×1 and 2×2 blocks on the main diagonal and can appear anywhere in the sequence (usually as S_1 or S_N). The product of conforming 1×1 and 2×2 diagonal blocks of the matrix sequence S_k gives the real and complex conjugated pairs of eigenvalues, respectively, of the matrix product $A_N\cdots A_2A_1$.

The main steps of the multi-shot method (Varga, 2005) applied to computing the stabilizing solution of the PRDE are:

(1) Compute the transition matrices $\Phi_N,\ldots,\Phi_2,\Phi_1$ by solving the initial value problem (5) for each interval $[k\Delta,(k-1)\Delta]$, for $k = 1,2,\ldots,N$.

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[2] In finite precision, computed eigenvalues may appear on or close to the boundary of the unit circle.

(2) Using the algorithm in (Bojanczyk *et al.*, 1992) compute the periodic real Schur form associated with the matrix product $\Psi(0) = \Phi_N \cdots \Phi_2\Phi_1$:

$$Z_{k+1}^T \Phi_k Z_k = S_k, \quad k = 1, \ldots, N, \quad (7)$$

with $Z_{N+1} = Z_1$ and S_1 quasi-triangular.

(3) Reorder the periodic real Schur form using the algorithm in (Granat and Kågström, 2006; Granat *et al.*, 2007) such that

$$Q_{k+1}^T S_k Q_k = \begin{bmatrix} \widetilde{S}_{11}^{(k)} & \widetilde{S}_{12}^{(k)} \\ 0 & \widetilde{S}_{22}^{(k)} \end{bmatrix}, \quad k = 1, \ldots, N, \quad (8)$$

with $Q_{N+1} = Q_1$ and where the matrix product $\widetilde{S}_{11}^{(N)} \cdots \widetilde{S}_{11}^{(1)}$ has n eigenvalues inside the unit circle, and $\widetilde{S}_{22}^{(N)} \cdots \widetilde{S}_{22}^{(1)}$ has n eigenvalues outside the unit circle. Here, Q_k is the sequence of orthogonal transformation matrices that perform the eigenvalue reordering of the PRSF (7).

(4) For each k, partition the product of the transformation matrices from (7) and (8) into $n \times n$ blocks as

$$Z_k Q_k = \begin{bmatrix} Y_{11}^{(k)} & Y_{12}^{(k)} \\ Y_{21}^{(k)} & Y_{22}^{(k)} \end{bmatrix}.$$

Then the solution of the PRDE at $t = (k-1)\Delta$, $k = 1, \ldots, N$, is

$$X((k-1)\Delta) = Y_{21}^{(k)} (Y_{11}^{(k)})^{-1}.$$

The multi-shot method has some important characteristics that are summarized below: (*i*) The ODE to compute $\Psi(0)$, which has unstable dynamics, is solved over short subparts of the period. Notably, these N ODEs can be solved independently, so this step is with favour solved in parallel; (*ii*) Only one ODE (in a multi-shot fashion) must be solved in sequence, in contrast to the one-shot method where two ODEs dependent on each other must be solved; (*iii*) The system's periodicity is exploited, by explicitly using methods designed for periodic systems. Altogether this makes it likely that the multi-shot method is more reliable which is investigated in the next section.

4. COMPUTATIONAL EXPERIMENTS

We evaluate and compare the one-shot method and the multi-shot method on two test problems. The first is an artificial time-varying system for which the exact solution can be computed, and the second problem is a devil stick model considered in (Nakaura *et al.*, 2004; Freidovich *et al.*, 2007; Freidovich *et al.*, 2006). The comparison is performed using both ordinary ODE methods and symplectic methods (Hairer *et al.*, 2006; Leimkuhler and Reich, 2004; McLachlan, 2007) for solving the Hamiltonian systems (5) and (6). The implementation of the two methods has been done in MATLAB, utilizing built-in functions and gateways to existing Fortran subroutines (notably, periodic eigenvalue reordering by (Granat and Kågström, 2006) and symplectic solvers by (Hairer *et al.*, 2006)).

In some of the figures (e.g., see Figure 1), solutions $X(t)$ of the PRDE are plotted componentwise for each discrete time $t = (k-1)\Delta$ where $k = 1, \ldots, N$, i.e., each curve in a plot corresponds to how one element in $X(t)$ evolves over time.

4.1 Artificial time-varying system

Consider a linear time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(t_0) = x_0, \quad (9)$$

with two states and two inputs, i.e., $A \in \mathbb{R}^{2\times 2}$ and $B \in \mathbb{R}^{2\times 2}$. It has the optimal feedback control

$$u^*(t) = -Kx(t), \text{ where } K = \Gamma^{-1}B^T X. \quad (10)$$

For linear time-invariant systems, X in the optimal feedback control (10) is obtained by solving the *algebraic Riccati equation* (ARE)

$$A^T X + XA - XB\Gamma^{-1}B^T X + Q = 0. \quad (11)$$

To solve (11) an existing stable solver is used (Arnold and Laub, 1984; Laub, 1979).

Then the time-invariant system (9) is transformed into a periodic time-varying system by change of coordinates: $z(t) = P(t)x(t)$ where

$$P(t) = \begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix},$$

for a suitable integer ω. This results in the T-periodic time-varying system

$$\dot{z}(t) = \widetilde{A}(t)z(t) + \widetilde{B}(t)u(t),$$

where

$$\widetilde{A}(t) = \frac{dP(t)}{dt}P(t)^{-1} + P(t)AP(t)^{-1}, \text{ and}$$
$$\widetilde{B}(t) = P(t)B,$$

with period $T = 2\pi/\omega$, and the weighting functions $\widetilde{Q}(t) = P(t)^{-T}QP(t)^{-1}$ and $\widetilde{\Gamma}(t) = \Gamma$. The optimal feedback is $u^*(t) = -K(t)z(t) = -\Gamma^{-1}\widetilde{B}(t)^T\widetilde{X}(t)z(t)$, where $\widetilde{X}(t)$ is the computed solution of the PRDE (4). The solution $X(t) = P(t)^{-T}XP(t)^{-1}$, where X is the solution of (11), corresponds to the exact solution at time t (our *reference solution*).

In the following, the *relative error* of the PRDE solution with respect to the reference solution is

$$\sum_{k=1}^{N} \left(\frac{\|\widetilde{X}_k - X_k\|_F}{\|X_k\|_F} \right) / N,$$

where $X_k = X((k-1)T/N)$, T is the periodicity, and N is the number of steps in the multi-shot PRDE solver.

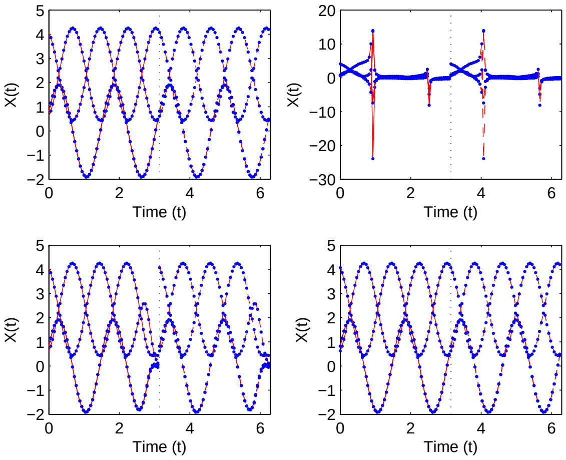

Fig. 1. Solutions of the PRDE for the artificial system over two periods using ODE45. (Top-left) Reference solution $X(t)$. (Top-right) One-shot solution with default tol. parameters for ODE45, ($RelTol = 10^{-3}$ and $AbsTol = 10^{-6}$). (Bottom-left) One-shot solution with $RelTol = 10^{-12}$ and $AbsTol = 10^{-16}$. (Bottom-right) Multi-shot solution with $N = 60$ and default tol. parameters for ODE45.

For the computational experiments, consider a linear time-invariant system with

$$A = \begin{bmatrix} 1 & 0.5 \\ 3 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ 1 \end{bmatrix},$$

and the weighting functions $Q = I_2$ and $\Gamma = 1$. The period for the corresponding periodic time-varying system is chosen to $T = \pi$ (i.e., $\omega = 2$).

In the first test, the general purpose variable step-size solver ODE45 in Matlab is used to solve the Hamiltonian systems (5) and (6). In Figure 1, it can be seen that the one-shot method does not result in an accurate periodic solution, even if the tolerance parameters of ODE45 are decreased to $RelTol = 10^{-12}$ and $AbsTol = 10^{-16}$. The multi-shot method on the other hand performs very well. The poor performance of the one-shot method is due to the use of a non-symplectic ODE solver over a long time period. As can be expected for the multi-shot method, the relative error of the solution decays with N, the number of time periods, see Figure 2. Note, when using ODE45 with the strict tolerance parameters, $RelTol = 10^{-12}$ and $AbsTol = 10^{-16}$, the computation time is drastically increased.

In the second test, the implicit *Gauss Runge-Kutta* (GRK) methods (Hairer *et al.*, 2006) of orders 4, 8, and 12 (i.e., 2, 4, and 6 stages) with fixed time steps are used to solve (5)–(6). These methods are designed to be structure preserving with respect to symplecticity and symmetry.

For the one-shot method, the best solution is obtained using the GRK solver of order 12 with 200 time steps. The solution is similar to the

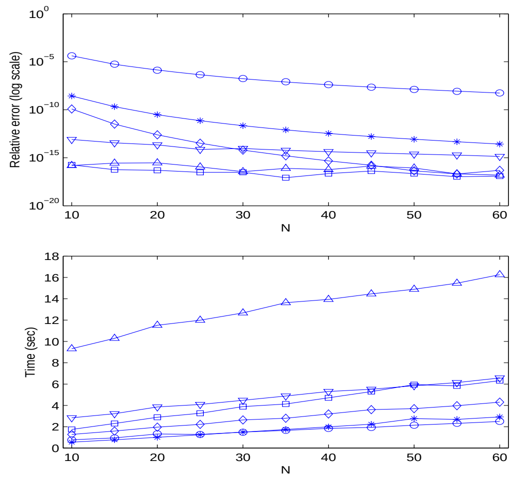

Fig. 2. The relative errors and the computation times of the multi-shot PRDE solutions for different values of N. ODE solvers used: ($*$) ODE45 (default tol. parameters), (∇) ODE45 ($RelTol = 10^{-9}$ and $AbsTol = 10^{-16}$), ($\triangle$) ODE45 ($RelTol = 10^{-12}$ and $AbsTol = 10^{-16}$), ($\square$) GRK of order 12, ($\diamond$) GRK of order 8, and ($\circ$) GRK of order 4.

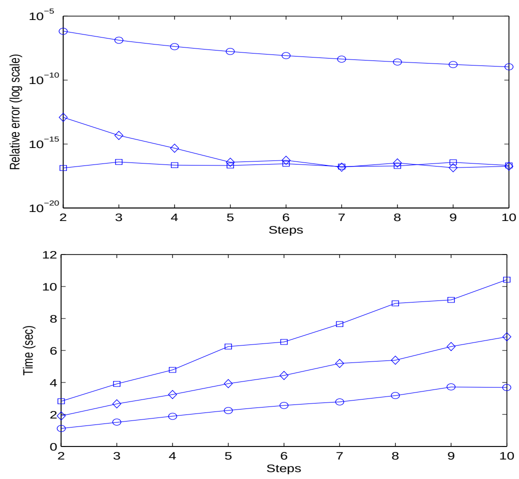

Fig. 3. The relative errors and the computation times of the multi-shot PRDE solutions for $N = 40$ computed with GRK using different number of time steps. Solver: ($\square$) order 12, ($\diamond$) order 8, and ($\circ$) order 4.

best solution computed with ODE45, i.e., the one-shot method still fails to compute an accurate periodic solution for the PRDE, see Figure 1 (bottom-left). This problem could probably be solved with another choice of time step method and/or symplectic ODE solver, e.g., see (Hairer *et al.*, 2006; McLachlan, 2007). Since the multi-shot method solves the Hamiltonian system over shorter time periods, it solves the problem accurately both using a symplectic solver and ODE45, see Figure 1 (bottom-right).

The relative errors for the multi-shot method with the symplectic GRK solvers using 4 time steps

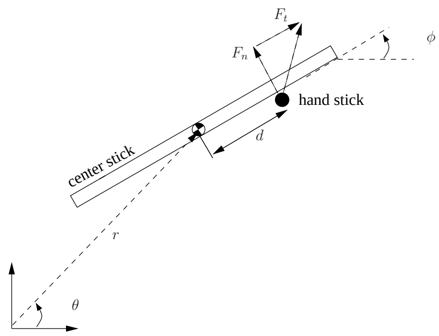

Fig. 4. Model of the devil stick (Freidovich *et al.*, 2007; Nakaura *et al.*, 2004).

are displayed in Figure 2. The improved accuracy which is acquired with a symplectic solver comes with an overhead of increased computation time, see lower graph. So the choice of method is a trade-off between the computational cost (efficiency) and the accuracy of the computed solution. For best accuracy in the solution, GRK of order 8 or 12 should be used. Already for $N = 10$, the solver of order 12 has reached the tolerance used in the solver. If a fast solver with moderate accuracy is wanted, either ODE45 with default tolerance parameters or GRK of order 8 is appropriate. The GRK method of order 4 performs worse than all the other solvers for any N. As can be seen in Figure 3, the results can slightly be improved by using more time steps but to the cost of an increasing amount of work. Note, the number of time steps for GRK of order 8 and 12 should be kept relative low, in this case below 5, since the tolerance of the solver is reached rather quickly but the computation time continues to increase with the number of time steps.

4.2 Devil stick model

The devil stick is a juggling device which consists of a center stick and a hand stick. The center stick has a periodic propeller-like motion which is induced by a contact force from the hand stick, see Figure 4.

The dynamics and the resulting stabilizing controller for the devil stick are just briefly described below. For further details, see (Freidovich *et al.*, 2007; Freidovich *et al.*, 2006; Nakaura *et al.*, 2004). The design of the stabilizing feedback controller is developed in (Freidovich *et al.*, 2007; Freidovich *et al.*, 2006), and from there we also choose model parameters of the devil stick.

The dynamics of the center stick, in polar coordinates, are (Nakaura *et al.*, 2004; Freidovich *et al.*, 2007) [3] :

$$
\begin{aligned}
\ddot{r} &= r\dot{\theta}^2 - g\sin(\theta) + \frac{\cos(\theta-\phi)}{m}F_t \\
&\quad + \frac{\sin(\theta-\phi)}{m}F_n, \\
\ddot{\theta} &= -\frac{2\dot{r}\dot{\theta}}{r} - \frac{g\cos(\theta)}{r} - \frac{\sin(\theta-\phi)}{rm}F_t \\
&\quad + \frac{\cos(\theta-\phi)}{rm}F_n, \\
\ddot{\phi} &= \frac{d(\phi)}{J}F_n = \frac{-\rho\phi + d_0}{J}F_n.
\end{aligned}
$$

The used model parameters of the devil stick are $m = 0.2$ [kg], $J = 0.01$ [kg m^2], $\rho = 0.03$ [m], $g = 9.81$ [kg/s^2], $r = 0.05$ [m], and $d_0 = \rho\pi$.

One of the main steps in the design of a stabilizing feedback for the devil stick, consists of solving the LQR problem for the periodic linear system

$$
\frac{d}{dt}\begin{bmatrix} I_* \\ y_{1*} \\ y_{2*} \\ \dot{y}_{1*} \\ \dot{y}_{2*} \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) & 0 & a_{15}(t) \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{A(t)} \begin{bmatrix} I_* \\ y_{1*} \\ y_{2*} \\ \dot{y}_{1*} \\ \dot{y}_{2*} \end{bmatrix} + \underbrace{\begin{bmatrix} b_{11}(t) & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{B(t)} \begin{bmatrix} v_{1*} \\ v_{2*} \end{bmatrix},
$$

where

$$
\begin{aligned}
a_{11}(t) &= rmd(\phi_*(t))\dot{\phi}_*(t)/J, \\
a_{12}(t) &= md(\phi_*(t))\dot{\phi}_*(t)^3/J, \\
a_{13}(t) &= rmd(\phi_*(t))\dot{\phi}_*(t)\ddot{\phi}_*(t)/J, \\
a_{15}(t) &= 2rmd(\phi_*(t))\dot{\phi}_*(t)^2/J, \text{ and} \\
b_{11}(t) &= -md(\phi_*(t))\dot{\phi}_*(t)/J,
\end{aligned}
$$

with $d(\phi_*(t)) = \rho\phi_*(t) + d_0$. The variables $\phi_*(t)$ and $\dot{\phi}_*(t)$ are the solution of the differential equation

$$
-\frac{J}{md(\phi(t))}\ddot{\phi}(t) + r\dot{\phi}(t)^2 + g\cos(\phi(t)) = 0,
$$

with initial conditions $\phi(0) = 0.5$ and $\dot{\phi}(0) = 0$.

It follows that the matrices $A(t)$ and $B(t)$ are T-periodic matrices with period $T = 2.854$. The stabilizing controller is now given via solving the LQR problem. As for the artificial time-varying system, we focus on solving the PRDE (4). The following constant weighting matrices has been

[3] (r, θ) are the polar coordinates of the mass center of the center stick, ϕ is the angle of the center stick, $d(\phi)$ is the instantaneous position at which the center stick and hand stick are in contact, $d_0 = d(0)$ is the initial contact position, ρ is the radius of the hand stick, m is the mass of the center stick, J is its moment of inertia, and F_t and F_n are tangential and normal components of the force induced by the hand stick to the center stick.

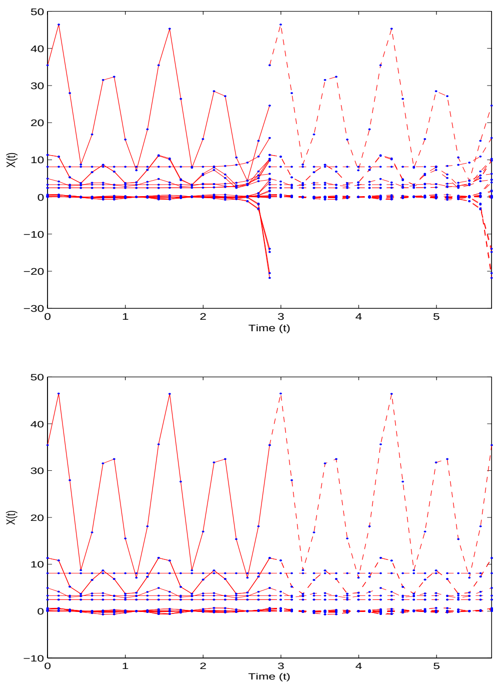

Fig. 5. The computed PRDE solution $X(t)$ for the devil stick plotted over two periods. (Top) The one-shot PRDE solver using MATLAB's ODE45. (Bottom) The multi-shot PRDE solver using the GRK solver of order 12 with 4 fixed time steps and $N = 20$.

used: $Q = \text{diag}\{0.004, 0.004, 6, 0.04, 6\}$ and $\Gamma = I_2$.

First the one-shot solver with MATLAB's ODE45 is tested. The PRDE solver does not preserve the periodic behavior of the system, see Figure 5. If instead the multi-shot method with $N = 20$ is used, still with MATLAB's ODE45, the solution $X(t)$ becomes periodic. Similar periodic results for the one-shot and multi-shot methods are also computed using the symplectic GRK method. So in this case, the one-shot method with a symplectic solver does produce robust periodic results, in contrast to the artificial time-varying system. In Figure 5, the computed periodic solution $X(t)$ is plotted for the multi-shot solver using the GRK of order 12. The computation times for the four cases are: One-shot with ODE45, 1min 25sec; One-shot with GRK, 14min 30sec; Multi-shot with ODE45, 12min 8sec; Multi-shot with GRK, 24min 39sec.

Future work includes further testings for deciding which of the four methods is best for the devil stick model and which model parameters to use.

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